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The Triangle Inequality

INTRODUCTION

A crucially important principle in analysis is the Triangle Inequality. This article will establish this principle "from the ground up."

Definition 1: The *absolute value* or *magnitude* of a real number a is denoted by "$|a|$" and is defined by

$$|a| = a \qquad \text{if } a \geq 0$$
$$|a| = -a \qquad \text{if } a < 0.$$

Theorem 1: $|a| \leq c$ if and only if for $-c \leq a \leq c$ for $a, c \in \mathbb{R}$.

Proof: Suppose that $|a| \leq c$. Either $a \geq 0$ or $a < 0$.

Case 1: Suppose that $a \geq 0$. Hence, $|a| = a$, by Definition 1. So, $a \leq c$, by substitution. Thus, $-a \geq -c$, viz., $-c \leq -a$. But $-a \leq a$. Consequently, $-c \leq a \leq c$.

Case 2: Suppose that $a < 0$. Hence, $|a| = -a$, by Definition 1. So, $-a \leq c$, by substitution. Thus, $a \geq -c$, viz., $-c \leq a$. But since $a < 0$, $a \leq -a$. Consequently, $-c \leq a \leq c$.

In either case, $-c \leq a \leq c$.

Suppose that $-c \leq a \leq c$. Either $a \geq 0$ or $a < 0$.

Case 1: Suppose that $a \geq 0$. Hence, $|a| = a$, by Definition 1. So, $-c \leq |a| \leq c$, by substitution. Thus, $|a| \leq c$.

Case 2: Suppose that $a < 0$. Hence, $|a| = -a$, by Definition 1. So, $-|a| = a$. Thus, $-c \leq -|a| \leq c$, by substitution. Hence, $c \geq |a| \geq -c$, viz., $-c \leq |a| \leq c$. So, $|a| \leq c$.

In either case, $|a| \leq c$.

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Theorem 2: $-|a| \leq a \leq |a|$ for $a \in \mathbb{R}$.

Proof: Either $a \geq 0$ or $a < 0$.

Case 1: Suppose that $a \geq 0$. Hence, $|a| = a$, by Definition 1. Furthermore, $-a \leq a \leq a$. So, $-|a| \leq a \leq |a|$, by substitution.

Case 2: Suppose that $a < 0$. Hence, $|a| = -a$, by Definition 1. So, $-|a| = a$. Furthermore, $a \leq a \leq -a$. Thus, $-|a| \leq a \leq |a|$, by substitution.

In either case, $-|a| \leq a \leq |a|$.

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Theorem 3 (The Triangle Inequality)**:** $|a + b| \leq |a| + |b|$ for any $a, b \in \mathbb{R}$.

Proof: $-|a| \leq a \leq |a|$ and $-|b| \leq b \leq |b|$, by Theorem 2. Hence, $-(|a| + |b|) \leq a + b \leq (|a| + |b|)$. So, $|a + b| \leq |a| + |b|$, by Theorem 1.

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