The Weekly Rigor

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Introduction to the Constant, Constant Multiple, Sum, and Difference Rules for Horizontal Asymptotes of Rational Functions
(Part 1)

INTRODUCTION

This article will lay out the proofs of the basic rules a student needs to algebraically determine the horizontal asymptote of a rational function. After the proofs, three solved problems will illustrate the use of the rules to address the three types of rational functions.

Theorem 1 (The Constant Rule)**:** $\lim_{x\to+\infty} c = c$ for any constant $c \in \mathbb{R}$.

Preliminary Remark: In words: The limit of a constant is equal to the constant.

Proof: Let $\varepsilon > 0$ be given and set $N = \varepsilon$. Hence, $N > 0$.

Suppose that $x > N$. Since $\varepsilon > 0$, $0 < \varepsilon$. $|c - c| = 0$ for any constant $c \in \mathbb{R}$. Hence, $|c - c| < \varepsilon$, by substitution. Consequently, if $x > N$, then $|c - c| < \varepsilon$.

Therefore, $\lim_{x\to+\infty} c = c$ for any constant $c \in \mathbb{R}$, by Definition 1 of *WR* no. 74.

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Theorem 2 (The Sum Rule)**:** If $\lim_{x\to+\infty} f(x) = L_1$ and $\lim_{x\to+\infty} g(x) = L_2$,
then $\lim_{x\to+\infty}[f(x) + g(x)] = L_1 + L_2$.

Preliminary Remark: In words: The limit of a sum is the sum of the limits.

Proof: Suppose that $\lim_{x\to+\infty} f(x) = L_1$ and $\lim_{x\to+\infty} g(x) = L_2$. Hence, since $\lim_{x\to+\infty} f(x) = L_1$, given any number $\varepsilon_1 > 0$, there corresponds a positive number N_1 such that if $x > N_1$, then $|f(x) - L_1| < \varepsilon_1$, by Definition 1 of *WR* no. 74. Furthermore, since $\lim_{x\to+\infty} g(x) = L_2$, given any number $\varepsilon_2 > 0$, there corresponds a positive number N_2 such that if $x > N_2$, then $|g(x) - L_2| < \varepsilon_2$, also by Definition 1 of *WR* no. 74.

Let $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ be given and set $\frac{\varepsilon}{2} = \max\{\varepsilon_1, \varepsilon_2\}$. Hence, there exists $N_1 > 0$ such that if $x > N_1$, then $|f(x) - L_1| < \varepsilon_1 \leq \frac{\varepsilon}{2}$, and there exists $N_2 > 0$ such that if $x > N_2$, then $|g(x) - L_2| < \varepsilon_2 \leq \frac{\varepsilon}{2}$. Set $N = \max\{N_1, N_2\}$.

Suppose that $x > N$. Hence, $x > N_1$ and $x > N_2$. So, $|f(x) - L_1| < \frac{\varepsilon}{2}$ and $|g(x) - L_2| < \frac{\varepsilon}{2}$. Thus, $|f(x) - L_1| + |g(x) - L_2| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. Hence, $|[f(x) - L_1] + [g(x) - L_2]| < \varepsilon$, by the Triangle Inequality (Cf. *WR* no. 75). So, $|[f(x) + g(x)] - [L_1 + L_2]| < \varepsilon$. Consequently, if $x > N$, then $|[f(x) + g(x)] - [L_1 + L_2]| < \varepsilon$.

Therefore, $\lim\limits_{x\to+\infty}[f(x) + g(x)] = L_1 + L_2$, by Definition 1 of *WR* no. 74.

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Theorem 3: If $\lim\limits_{x\to+\infty} f(x) = L$ then $\lim\limits_{x\to+\infty} cf(x) = cL$, for $c \in \mathbb{R}\backslash\{0\}$.

Proof: Suppose that $\lim\limits_{x\to+\infty} f(x) = L$. Let $\varepsilon > 0$ and $c \in \mathbb{R}\backslash\{0\}$ be given. Hence, since $\lim\limits_{x\to+\infty} f(x) = L$, there exists $N^* > 0$ such that if $x > N^*$, then $|f(x) - L| < \frac{\varepsilon}{|c|}$, by Definition 1 of *WR* no. 74. Set $N = N^*$.

Suppose that $x > N$. Hence, $x > N^*$. So, $|f(x) - L| < \frac{\varepsilon}{|c|}$. Thus, $|c||f(x) - L| < |c|\frac{\varepsilon}{|c|} = \varepsilon$. But $|c||f(x) - L| = |cf(x) - cL|$. Hence, $|cf(x) - cL| < \varepsilon$, by substitution. Consequently, if $x > N$, then $|cf(x) - cL|$.

Therefore, $\lim\limits_{x\to+\infty} cf(x) = cL$, by Definition 1 of *WR* no. 74.

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Theorem 4: If $\lim\limits_{x\to+\infty} f(x) = L$ then $\lim\limits_{x\to+\infty} cf(x) = cL$, for $c = 0$.

Proof: Suppose that $\lim\limits_{x\to+\infty} f(x) = L$ and $c = 0$. Hence, $\lim\limits_{x\to+\infty} cf(x) = \lim\limits_{x\to+\infty}[0 \cdot f(x)] = = \lim\limits_{x\to+\infty} 0 = 0 = 0 \cdot L = cL$, by Theorem 1.

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Theorem 5 (The Constant Multiple Rule)**:**

If $\lim\limits_{x\to+\infty} f(x) = L$ then $\lim\limits_{x\to+\infty} cf(x) = cL$, for $c \in \mathbb{R}$.

Proof: By Theorems 3 and 4.

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