The Weekly Rigor

No. 77 "A mathematician is a machine for turning coffee into theorems." December 12, 2015

Introduction to the Constant, Constant Multiple, Sum, and Difference Rules for Horizontal Asymptotes of Rational Functions

(Part 2)

Theorem 6 (The Difference Rule)**:** If $\lim_{x\to+\infty} f(x) = L_1$ and $\lim_{x\to+\infty} g(x) = L_2$, then $\lim_{x\to+\infty}[f(x) - g(x)] = L_1 - L_2$.

Preliminary Remark: In words: The limit of a difference is the difference of the limits.

Proof: Suppose that $\lim_{x\to+\infty} f(x) = L_1$ and $\lim_{x\to+\infty} g(x) = L_2$. Hence, $\lim_{x\to+\infty}[f(x) - g(x)] =$

$$= \lim_{x\to+\infty}[f(x) + (-1)g(x)] \overset{\text{T2}}{=} \lim_{x\to+\infty} f(x) + \lim_{x\to+\infty}(-1)g(x) \overset{\text{T5}}{=} L_1 + (-1)L_2 = L_1 - L_2.$$

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APPLICATIONS

In the context of horizontal asymptotes, there are three types of rational functions. The types are categorized on the basis of whether the degree of the denominator is greater than that of the numerator ("bottom heavy"), the degree of both the denominator and numerator are the same ("even heavy"), or the degree of the denominator is less than that of the numerator ("top heavy"). The following examples will involve finding horizontal asymptotes by applying the principles from Theorem 4 of *WR* no. 74 and this article.

1. "Bottom heavy" rational function.

$$\frac{2x^3+4x^2-5x+3}{7x^4-6x^3+2x+9} = \frac{(2x^3+4x^2-5x+3)}{(7x^4-6x^3+2x+9)} \cdot \frac{\left(\frac{1}{x^4}\right)}{\left(\frac{1}{x^4}\right)} = \frac{\frac{2x^3}{x^4}+\frac{4x^2}{x^4}-\frac{5x}{x^4}+\frac{3}{x^4}}{\frac{7x^4}{x^4}-\frac{6x^3}{x^4}+\frac{2x}{x^4}+\frac{9}{x^4}} = \frac{\frac{2}{x}+\frac{4}{x^2}-\frac{5}{x^3}+\frac{3}{x^4}}{7-\frac{6}{x}+\frac{2}{x^3}+\frac{9}{x^4}}.$$

Hence, $\lim_{x\to+\infty} \frac{\frac{2}{x}+\frac{4}{x^2}-\frac{5}{x^3}+\frac{3}{x^4}}{7-\frac{6}{x}+\frac{2}{x^3}+\frac{9}{x^4}} =$

$$= \frac{\lim_{x\to+\infty}\left[\frac{2}{x}+\frac{4}{x^2}-\frac{5}{x^3}+\frac{3}{x^4}\right]}{\lim_{x\to+\infty}\left[7-\frac{6}{x}+\frac{2}{x^3}+\frac{9}{x^4}\right]} = \frac{\lim_{x\to+\infty}\frac{2}{x}+\lim_{x\to+\infty}\frac{4}{x^2}-\lim_{x\to+\infty}\frac{5}{x^3}+\frac{3}{x^4}}{\lim_{x\to+\infty}7-\lim_{x\to+\infty}\frac{6}{x}+\lim_{x\to+\infty}\frac{2}{x^3}+\lim_{x\to+\infty}\frac{9}{x^4}} = \frac{0+0-0+0}{7-0+0+0} = \frac{0}{7} = 0.$$

Since all bottom heavy rational functions end up with a 0 in the numerator and some other number in the denominator, they all have $y = 0$ as their horizontal asymptote.

2. “Even heavy” rational function.

$\frac{2x^4+4x^2-5x+3}{7x^4-6x^3+2x+9} = \frac{(2x^4+4x^2-5x+3)}{(7x^4-6x^3+2x+9)} \cdot \frac{\left(\frac{1}{x^4}\right)}{\left(\frac{1}{x^4}\right)} = \frac{\frac{2x^4}{x^4}+\frac{4x^2}{x^4}-\frac{5x}{x^4}+\frac{3}{x^4}}{\frac{7x^4}{x^4}-\frac{6x^3}{x^4}+\frac{2x}{x^4}+\frac{9}{x^4}} = \frac{2+\frac{4}{x^2}-\frac{5}{x^3}+\frac{3}{x^4}}{7-\frac{6}{x}+\frac{2}{x^3}+\frac{9}{x^4}}$. Hence, $\lim_{x\to+\infty} \frac{2+\frac{4}{x^2}-\frac{5}{x^3}+\frac{3}{x^4}}{7-\frac{6}{x}+\frac{2}{x^3}+\frac{9}{x^4}} =$

$$= \frac{\lim_{x\to+\infty}\left[2+\frac{4}{x^2}-\frac{5}{x^3}+\frac{3}{x^4}\right]}{\lim_{x\to+\infty}\left[7-\frac{6}{x}+\frac{2}{x^3}+\frac{9}{x^4}\right]} = \frac{\lim_{x\to+\infty} 2+\lim_{x\to+\infty}\frac{4}{x^2}-\lim_{x\to+\infty}\frac{5}{x^3}+\frac{3}{x^4}}{\lim_{x\to+\infty} 7-\lim_{x\to+\infty}\frac{6}{x}+\lim_{x\to+\infty}\frac{2}{x^3}+\lim_{x\to+\infty}\frac{9}{x^4}} = \frac{2+0-0+0}{7-0+0+0} = \frac{2}{7}.$$

Since all even heavy rational functions end up with the numerator’s leading coefficient in the numerator and the denominator’s leading coefficient in the denominator, they all have $y = \frac{a}{b}$ as their horizontal asymptote, where a is the numerator’s leading coefficient and b is the denominator’s leading coefficient.

3. “Top heavy” rational function.

$\frac{2x^5+4x^2-5x+3}{7x^4-6x^3+2x+9} = \frac{(2x^5+4x^2-5x+3)}{(7x^4-6x^3+2x+9)} \cdot \frac{\left(\frac{1}{x^4}\right)}{\left(\frac{1}{x^4}\right)} = \frac{\frac{2x^5}{x^4}+\frac{4x^2}{x^4}-\frac{5x}{x^4}+\frac{3}{x^4}}{\frac{7x^4}{x^4}-\frac{6x^3}{x^4}+\frac{2x}{x^4}+\frac{9}{x^4}} = \frac{2x+\frac{4}{x^2}-\frac{5}{x^3}+\frac{3}{x^4}}{7-\frac{6}{x}+\frac{2}{x^3}+\frac{9}{x^4}}$. Hence,

$$= \lim_{x\to+\infty} \frac{2x+\frac{4}{x^2}-\frac{5}{x^3}+\frac{3}{x^4}}{7-\frac{6}{x}+\frac{2}{x^3}+\frac{9}{x^4}} = \frac{\lim_{x\to+\infty}\left[2x+\frac{4}{x^2}-\frac{5}{x^3}+\frac{3}{x^4}\right]}{\lim_{x\to+\infty}\left[7-\frac{6}{x}+\frac{2}{x^3}+\frac{9}{x^4}\right]} = \frac{\lim_{x\to+\infty} 2x+\lim_{x\to+\infty}\frac{4}{x^2}-\lim_{x\to+\infty}\frac{5}{x^3}+\frac{3}{x^4}}{\lim_{x\to+\infty} 7-\lim_{x\to+\infty}\frac{6}{x}+\lim_{x\to+\infty}\frac{2}{x^3}+\lim_{x\to+\infty}\frac{9}{x^4}} = \frac{\infty+0-0+0}{7-0+0+0} = \frac{\infty}{7} = +\infty.$$

Since all top heavy rational functions grow without any upper bound, they all fail to have any horizontal asymptote.