The Weekly Rigor

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Differentiability Implies Continuity, but Not Vice-Versa

(Part 1)

INTRODUCTION

Early in the study of differential calculus, the student will be exposed to a principle uniting the concepts of “differentiability” and “continuity,” viz., that the former necessitates the latter. A common mistake for the new student is to take the message that continuity necessitates differentiability. The accurate principle is that *differentiability implies continuity, but not vice-versa*, the proof of which is the subject of this article. Textbooks differ on how they prove the claim that differentiability implies continuity, so I have included two different proofs to cover some of the possible varieties.

Definition 1: A function *f* is said to be *continuous at a point a* if the following condition is satisfied:

$$\lim_{x \to a} f(x) = f(a).$$

Definition 2: The function f' defined by the formula

$$f'(x) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

is called *the derivative with respect to* x of the function *f*. The domain of f' consists of all *x* for which the limit exists.

Theorem 1 (Differentiability Implies Continuity, First Proof)**:**

If *f* is differentiable at a point a, then *f* is also continuous at a.

Proof: Suppose that *f* is differentiable at a point a. Hence, $\lim_{x \to a}[f(x) - f(a)] =$

$= \lim_{x \to a}\left[\frac{f(x)-f(a)}{x-a} \cdot \frac{x-a}{1}\right] = \lim_{x \to a}\frac{f(x)-f(a)}{x-a} \cdot \lim_{x \to a}(x - a) \overset{\text{D2}}{=} f'(x) \cdot 0 = 0$. So, $\lim_{x \to a} f(x) - \lim_{x \to a} f(a) = \lim_{x \to a} f(x) - f(a) = 0$. Thus, $\lim_{x \to a} f(x) = f(a)$. Therefore, *f* is continuous at point a, by Definition 1.

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Theorem 2: A function *f* is continuous at a point *a* if the following condition is satisfied:

$$\lim_{h\to 0} f(a+h) = f(a).$$

Proof: Suppose that

$$\lim_{h\to 0} f(a+h) = f(a).$$

Set $h = x - a$. Hence, $a + h = x$ and $x \to a$ as $h \to 0$. So,

$$\lim_{x\to a} f(x) = f(a),$$

by substitution. Therefore, *f* is continuous at point *a*, by Definition 1.

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Theorem 3: $$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}.$$

Proof: Set $x = a + h$. Hence, $x - a = h$ and $h \to 0$ as $x \to a$. So, $f'(x) \overset{\text{D2}}{=} \lim_{x\to a} \frac{f(x)-f(a)}{x-a} \overset{\text{SUB}}{=}$
$\overset{\text{SUB}}{=} \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$.

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Theorem 4 (Differentiability Implies Continuity, Second Proof)**:**

If *f* is differentiable at a point *a*, then *f* is also continuous at *a*.

Proof: Suppose that *f* is differentiable at a point *a*. Hence, $\lim_{h\to 0}[f(a+h) - f(a)] =$
$= \lim_{h\to 0} \left[\frac{f(a+h)-f(a)}{h} \cdot \frac{h}{1}\right] = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} \cdot \lim_{h\to 0} h \overset{\text{T3}}{=} f'(a) \cdot 0 = 0$. So,
$\lim_{h\to 0} f(a+h) - \lim_{h\to 0} f(a) = \lim_{h\to 0} f(x) - f(a) = 0$. Thus, $\lim_{h\to 0} f(a+h) = f(a)$. Therefore,
f is continuous at point *a*, by Theorem 2.

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