The Weekly Rigor

No. 79 | "A mathematician is a machine for turning coffee into theorems." | December 26, 2015

Differentiability Implies Continuity, but Not Vice-Versa
(Part 2)

To demonstrate why continuity does not imply differentiability, consider the function $f(x) = |x|$ at the point $x = 0$. Recall the definition of this function in *WR* no. 75:

The *absolute value* or *magnitude* of a real number a is denoted by "$|a|$" and is defined by

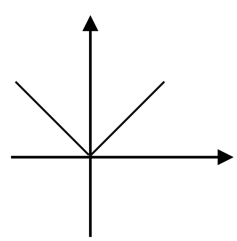
$$|a| = a \qquad \text{if } a \geq 0$$
$$|a| = -a \qquad \text{if } a < 0.$$

First, let us consider the proof of why $f(x) = |x|$ is continuous at $x = 0$. Note that $\lim_{x\to 0^-}|x| = \lim_{x\to 0^-}(-x) = 0$, and $\lim_{x\to 0^+}|x| = \lim_{x\to 0^+}x = 0$. Hence, $\lim_{x\to 0}|x| = 0$, by the Two-Sided Limit Test (Cf. *WR* no. 69). Furthermore, $f(0) = |0| = 0$. So, $\lim_{x\to 0}|x| = |0|$. Thus, f is continuous at $x = 0$, by Definition 1.

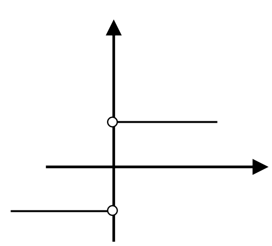
Now let us consider whether f is differentiable at $x = 0$. Observe that $\lim_{x\to 0^-}\frac{|x|-|0|}{x-0} = = \lim_{x\to 0^-}\frac{|x|}{x} = \lim_{x\to 0^-}\frac{-x}{x} = \lim_{x\to 0^-}(-1) = -1$. Furthermore, $\lim_{x\to 0^+}\frac{|x|-|0|}{x-0} = \lim_{x\to 0^+}\frac{|x|}{x} = \lim_{x\to 0^+}\frac{x}{x} = = \lim_{x\to 0^+}1 = 1$. Hence, $\lim_{x\to 0^-}\frac{|x|-|0|}{x-0} \neq \lim_{x\to 0^+}\frac{|x|-|0|}{x-0}$. So, $\lim_{x\to 0}\frac{|x|-|0|}{x-0}$ does not exist, by the Two-Sided Limit Test. Thus, $f'(0)$ does not exist (is not defined), by Definition 2.

The graphs of f and f' near the origin illustrate that the derivative does not exist at $x = 0$ even though $f(0)$ does exist. Note the sharp corner of the graph of f at the origin.

f:



f':



To sum up the above discussion, we have proved that if a function is differentiable at a point, then it is also continuous there; however, it is not true that if a function is continuous at a point, then it is also necessarily differentiable there. Symbolically, $\mathrm{D} \Rightarrow \mathrm{C}$ but $\mathrm{C} \nRightarrow \mathrm{D}$, i.e., the converses are not logically equivalent. As an analogy, if you know a dog is in the house, you know an animal is in the house; however if all you know is that there is an animal is in the house, you cannot deduce that the animal is a dog (it may be a cat).

To keep the proper order in mind, viz., $\mathrm{D} \Rightarrow \mathrm{C}$ but *not* $\mathrm{C} \Rightarrow \mathrm{D}$, one may recall the phrase "AC/DC" or "Washington, D.C."

Furthermore, although converse statements are not logically equivalent, *contrapositives* are. Hence, we may rightfully say that if a function is *not* continuous at a point, then it is also *not* differentiable there. Symbolically, $\sim\mathrm{C} \Rightarrow \sim\mathrm{D}$.

Finally it should be noted that the importance of being mindful of converse statements in calculus arises again in a big way in the study of series convergence tests (specifically, the "*n*th-term test" also known as the "Divergence Test").