The Weekly Rigor

No. 80 "A mathematician is a machine for turning coffee into theorems." January 2, 2016

Ten Factoring Principles for Elementary Algebra

In the following, let a, b, c, d, x, and y stand for any real numbers.

Theorem 1 (Difference of Squares)**:** $a^2 - b^2 = (a+b)(a-b)$.

Proof: $(a+b)(a-b) = (a+b)a - (a+b)b = a^2 + ab - ab - b^2 = a^2 - b^2$.

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Theorem 2 (Sum/Difference of Cubes)**:** $a^3 \pm b^3 = (a \pm b)(a^2 \mp ab + b^2)$.

Proof: $(a+b)(a^2 - ab + b^2) = (a+b)a^2 - (a+b)ab + (a+b)b^2 =$
$= a^3 + a^2b - a^2b - ab^2 + ab^2 + b^3 = a^3 + b^3$. Furthermore, $(a-b)(a^2 + ab + b^2) =$
$= (a-b)a^2 + (a-b)ab + (a-b)b^2 = a^3 - a^2b + a^2b - ab^2 + ab^2 - b^3 = a^3 - b^3$.
Therefore, $a^3 \pm b^3 = (a \pm b)(a^2 \mp ab + b^2)$.

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Theorem 3 (Perfect Square Trinomial)**:** $a^2 \pm 2ab + b^2 = (a \pm b)^2$.

Proof: $(a+b)^2 = (a+b)(a+b) = (a+b)a + (a+b)b = a^2 + ab + ab + b^2 =$
$= a^2 + 2ab + b^2$. Furthermore, $(a-b)^2 = (a-b)(a-b) = (a-b)a - (a-b)b =$
$= a^2 - ab - ab + b^2 = a^2 - 2ab + b^2$. Therefore, $a^2 \pm 2ab + b^2 = (a \pm b)^2$.

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Theorem 4: $a^3 \pm 3a^2b + 3ab^2 \pm b^3 = (a \pm b)^3$.

Proof: $(a+b)^3 = (a+b)(a+b)^2 \overset{\text{T3}}{=} (a+b)(a^2 + 2ab + b^2) =$
$= (a+b)a^2 + (a+b)2ab + (a+b)b^2 = a^3 + a^2b + 2a^2b + 2ab^2 + ab^2 + b^3 =$
$= a^3 + 3a^2b + 3ab^2 + b^3$. Furthermore, $(a-b)^3 = (a-b)(a-b)^2 \overset{\text{T3}}{=}$
$\overset{\text{T3}}{=} (a-b)(a^2 - 2ab + b^2) = (a-b)a^2 - (a-b)2ab + (a-b)b^2 =$
$= a^3 - a^2b - 2a^2b + 2ab^2 + ab^2 - b^3 = a^3 - 3a^2b + 3ab^2 - b^3$. Therefore,
$a^3 \pm 3a^2b + 3ab^2 \pm b^3 = (a \pm b)^3$.

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Theorem 5: $ax + ay \pm bx + by = (a \pm b)(x + y).$

Proof: $(a + b)(x + y) = (a + b)x + (a + b)y = ax + bx + ay + by = ax + ay + bx + by.$ Furthermore, $(a - b)(x + y) = (a - b)x + (a - b)y = ax - bx + ay - by =$
$= ax + ay - bx - by.$ Therefore, $ax + ay \pm bx + by = (a \pm b)(x + y).$

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Theorem 6: $x^2 + (a + b)x + ab = (x + a)(x + b).$

Proof: $(x + a)(x + b) = (x + a)x + (x + a)b = x^2 + ax + bx + ab = x^2 + (a + b)x + ab.$

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Theorem 7: $acx^2 + (ad + bc)x + bd = (ax + b)(cx + d).$

Proof: $(ax + b)(cx + d) = (ax + b)cx + (ax + b)d = acx^2 + bcx + adx + bd =$
$= acx^2 + adx + bcx + bd = acx^2 + (ad + bc)x + bd.$

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Theorem 8: $a^2 + 2ab + b^2 - c^2 = (a + b + c)(a + b - c).$

Proof: $(a + b + c)(a + b - c) = (a + b + c)a + (a + b + c)b - (a + b + c)c =$
$= a^2 + ab + ac + ab + b^2 + bc - ac - bc - c^2 = a^2 + 2ab + b^2 - c^2.$

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Theorem 9: $(ax - by)^2 + (bx + ay)^2 = (a^2 + b^2)(x^2 + y^2).$

Proof: $(a^2 + b^2)(x^2 + y^2) = (a^2 + b^2)x^2 + (a^2 + b^2)y^2 = a^2x^2 + b^2x^2 + a^2y^2 + b^2y^2 =$
$= a^2x^2 + b^2y^2 + b^2x^2 + a^2y^2 = a^2x^2 - 2abxy + b^2y^2 + b^2x^2 + 2abxy + a^2y^2 \stackrel{\text{T3}}{=}$
$\stackrel{\text{T3}}{=} (ax - by)^2 + (bx + ay)^2.$

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Theorem 10: $(a^2 - b^2)^2 + (2ab)^2 = (a^2 + b^2)^2.$

Proof: $(a^2 + b^2)^2 = (a^2 + b^2)(a^2 + b^2) = (a^2 + b^2)a^2 + (a^2 + b^2)b^2 =$
$= a^4 + a^2b^2 + a^2b^2 + b^4 = a^4 + 2a^2b^2 + b^4 = a^4 - 2a^2b^2 + b^4 + 4a^2b^2 \stackrel{\text{T3}}{=}$
$\stackrel{\text{T3}}{=} (a^2 - b^2)^2 + (2ab)^2.$

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