The Weekly Rigor

No. 81 "A mathematician is a machine for turning coffee into theorems." January 9, 2016

An Essential Skill for Calculus Students: Factoring and Expanding

(Part 1)

The quantities which, when added (or subtracted) together, produce a quantity, are called the *terms* of the quantity. For example, x^2, $5x$, and 6 are the terms of $x^2 + 5x - 6$. The quantities which, when multiplied together, produce a quantity, are called the *factors* of the quantity. For example, 5 and x are the factors of $5x$. Keeping the conceptual distinction between "terms" and "factors" is crucial to the mathematics student since the properties of addition and multiplication have important algebraic differences.

Factoring is the process of splitting an expression of two or more terms into an equivalent product of two or more factors. For example, $x^2 + 5x - 6$ can be factored into $(x + 6)(x - 1)$. There are five factoring principles the calculus student must know: 1. Greatest Common Factor; 2. Difference of Squares; 3. Sum/Difference of Cubes; 4. Perfect Square Trinomials; 5. Completing the Square.

1. Greatest Common Factor. Always be on the lookout for terms that have a common factor. This factoring technique is very common and powerful. For example, the GCF of $4x^3 + 6x^2 - 10x$ is $2x$. Hence, when factored we have $2x(2x^2 + 3x - 5)$. Even if you fail to identify the *greatest* common factor, getting the expression into any sort of factored form is a step in a right direction.

2. Difference of Squares. This is the next most important factoring principle. This technique can be summarized as

$$a^2 - b^2 = (a + b)(a - b)$$

Examples:

$$x^2 - 1 = x^2 - 1^2 = (x + 1)(x - 1)$$

$$x^6 - x^4 = x^4(x^2 - 1) = x^4(x + 1)(x - 1)$$

$$9e^{4x} - 4e^{2x} = e^{2x}(9e^{2x} - 4) = e^{2x}((3e^x)^2 - 2^2) = e^{2x}(3e^x + 2)(3e^x - 2)$$

$$(x + y)^2 - z^2 = \big((x + y) + z\big)\big((x + y) - z\big)$$

3. Sum/Difference of Cubes. These factoring techniques do not come very often, but when they do they can blindside the unprepared and make for an unpleasant day. They are:

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$$

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

Examples:

$$x^3 + 8 = x^3 + 2^3 = (x + 2)(x^2 - x \cdot 2 + 2^2) = (x + 2)(x^2 - 2x + 4)$$

$$e^{3x} - 27 = (e^x)^3 - 3^3 = (e^x - 3)((e^x)^2 + e^x \cdot 3 + 3^2) = (e^x - 3)(e^{2x} + 3e^x + 9)$$

$$125 - 8x^3 = 5^3 - (2x)^3 = (5 - 2x)(5^2 + 5 \cdot 2x + (2x)^2) = (5 - 2x)(25 + 10x + 4x^2)$$

4. Perfect Square Trinomials. These factoring principles are important to keep in mind, for they can be crucial to overcome an algebraic logjam. They are:

$$a^2 + 2ab + b^2 = (a + b)^2$$

$$a^2 - 2ab + b^2 = (a - b)^2$$

Think of these principles especially when you have a square root you would like to eliminate. For example, $\sqrt{x^2 + 10x + 25} = \sqrt{(x + 5)^2} = |x + 5|$ and $\sqrt{e^{2x} + 8e^x + 16} = \sqrt{(e^x + 4)^2} =$ $= e^x + 4$. One method for creating a perfect square trinomial is Completing the Square.

5. Completing the Square. There are two types of this factoring procedure: a.) the leading coefficient is equal to 1; b.) the leading coefficient is not equal to 1.

Examples:

$$x^2 + 4x + 7 \;=\; (x^2 + 4x + 2^2) + 7 - 2^2 \;=\; (x + 2)^2 + 7 - 4 \;=\; (x + 2)^2 + 3$$

$$2x^2 + 4x + 7 \;=\; 2(x^2 + 2x) + 7 \;=\; 2(x^2 + 2x + 1^2) + 7 - 2 \;=\; 2(x + 1)^2 + 5$$

There are two tricky aspects to the second type: Remember to factor out the value of the leading coefficient from the first *two* terms, and be careful about what quantity you need to subtract back to preserve equality.