The Weekly Rigor

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An Essential Skill for Calculus Students: Factoring and Expanding

(Part 2)

The opposite process of factoring is expanding. *Expanding* is the process of multiplying out a factored expression into an equivalent series of terms. For example, $(x+6)(x-1)$ can be multiplied out by the "FOIL" method to $x^2 - x + 6x - 6$, which can be simplified to $x^2 + 5x - 6$. There are three additional expansion principles the calculus student should know: 2. Product of Binomial Conjugates; 3. Squared Binomials; 4. Cubed Binomials.

2. Product of Binomial Conjugates. This expansion method is the reverse of the factoring principle "Difference of Squares." Hence, in general, the pattern is

$$(a+b)(a-b) = a^2 - b^2$$

I suggest the student automatize this pattern without going through the intermediate labor of using the "FOIL" method. In words, the pattern would be: Write down the square of the first term, a^2, and subtract from that the square of the second term,
$a^2 - b^2$.

Examples:

$$(x+1)(x-1) = x^2 - 1$$

$$(3x+5)(3x-5) = 9x^2 - 25$$

$$(4e^{2x} + 2e^{6x})(4e^{2x} - 2e^{6x}) = 16e^{4x} - 4e^{12x}$$

$$\big((x+y)+z\big)\big((x+y)-z\big) = (x+y)^2 - z^2$$

3. Squared Binomials. This expansion method is the opposite of the factoring principle Perfect Square Trinomials. The two general patterns of this expansion technique can be stated as

$$(a+b)^2 = a^2 + 2ab + b^2$$

or

$$(a-b)^2 = a^2 - 2ab + b^2$$

The student can write a squared binomial such as $(a+b)^2$ as a product of two binomials $(a+b)(a+b)$ and perform the "FOIL" method on the latter. However, I think a shorthand method is advisable in this case, since the squared binomial is a very common mathematical expression. For the first pattern, the shorthand method consists of taking the given squared binomial $(a+b)^2$ and writing out the square of the first term, a^2, adding it to *double* the product of the two terms, a^2+2ab, and then adding to the latter the square of the last term, $a^2+2ab+b^2$. The shorthand method for the second pattern is the same, except that one subtracts the double of the product of the two terms.

Examples:

$$(x+1)^2 = x^2+2x+1$$

$$(x-3)^2 = x^2-6x+9$$

$$(x^3+4)^2 = x^6+8x^3+16$$

$$(3e^{2x}-e^{5x})^2 = 9e^{4x}-6e^{7x}+e^{10x}$$

$$\big((x+y)+z\big)^2 = (x+y)^2+2(x+y)z+z^2$$

4. Cubed Binomials. There is a principle called the "binomial formula" to immediately expand expressions of the form $(a+b)^3$ or $(a-b)^3$. However, I never found memorizing the binomial formula to be worth the time. Expanding cubed binomials is a rarity and there is an easy optional method: "FOIL" followed by multiplying by $(a \pm b)$. That is:

$$(a+b)^3 = (a+b)^2(a+b) = (a^2+2ab+b^2)(a+b) =$$
$$= a^3+2a^2b+ab^2+a^2b+2ab^2+b^3 = a^3+3a^2b+3ab^2+b^3$$

and

$$(a-b)^3 = (a-b)^2(a-b) = (a^2-2ab+b^2)(a-b) =$$
$$= a^3-2a^2b+ab^2-a^2b+2ab^2-b^3 = a^3-3a^2b+3ab^2-b^3$$

Summary Table of the Factoring and Expanding Methods

Factoring	Expanding
1. Greatest Common Factor	1. FOIL
2. Difference of Squares	2. Product of Binomial Conjugates
3. Sum/Difference of Cubes	3. Squared Binomials
4. Perfect Square Trinomials	4. Cubed Binomials
5. Completing the Square	