The Weekly Rigor

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An Essential Skill for Calculus Students: Fractions

Dealing with fractional expressions is very common in calculus. There are two broad topics regarding fractions that the new calculus student should be aware of: 1. the basic algebraic rules and manipulations; 2. Syntax options.

1. The basic algebraic rules and manipulations. The calculus student must have the basic algebra rules for fractions thoroughly automatized. (The rules discussed here pertain only to simple and complex fractions: “Mixed” numbers such as $4\frac{2}{3}$ do not arise in calculus.) To add two fractions, the denominators must be equal and one just adds the numerators, e.g., $\frac{2}{3}+\frac{x}{3}=\frac{2+x}{3}$. To multiply two fractions, one just multiples separately the numerators and denominators, e.g., $\frac{5}{8}\cdot\frac{x}{2}=\frac{5x}{16}$. If one wishes to add two fractions with unequal denominators, the fractions must be made to have identical denominators, e.g., $\frac{2}{3}+\frac{x}{4}=\frac{4}{4}\cdot\frac{2}{3}+\frac{x}{4}\cdot\frac{3}{3}=\frac{8}{12}+\frac{3x}{12}=\frac{8+3x}{12}$. To divide two fractions, multiply the top fraction by the reciprocal of the bottom fraction, e.g., $\frac{\frac{2x}{9}}{\frac{5}{7}}=\frac{2x}{9}\cdot\frac{7}{5}=\frac{14x}{45}$. To divide a fraction by an integer, such as $\frac{\left(\frac{4x}{5}\right)}{3}$, make the integer an explicitly equivalent fraction, viz., $\frac{\frac{4x}{5}}{\frac{3}{1}}$, and then proceed to compute, as $\frac{4x}{5}\cdot\frac{1}{3}=\frac{4x}{15}$. To divide an integer by a fraction, such as $\frac{3}{\left(\frac{4x}{5}\right)}$, make the integer an explicitly equivalent fraction, viz., $\frac{\frac{3}{1}}{\frac{4x}{5}}$, and then carry out the further computations, i.e., $\frac{3}{1}\cdot\frac{5}{4x}=\frac{15}{4x}$.

One situation that often arises in calculus is the need to reduce a complex fraction to a simple fraction. In arithmetic, a *simple fraction* is a rational number of the form $\frac{a}{b}$, where both a and b are integers ($b \neq 0$), e.g., $\frac{2}{3}$ and $\frac{8}{5}$. In algebra, a simple fraction can include examples with variables such as $\frac{2x}{3}$, where neither the numerator nor the denominator contains fractions. On the other hand, a *complex fraction* is a fraction whose numerator and/or denominator *does* contains fraction(s), e.g., $\frac{\left(\frac{7}{8}\right)}{15}$, $\frac{\frac{6}{5x}}{\frac{3}{8}}$, $\frac{\frac{3x}{5}+\frac{2}{3}}{6}$, $\frac{\frac{3}{5}+2}{\frac{5}{2}+\frac{22}{3x}}$. If the complex fraction contains no terms, as is the case for the first two examples immediately preceding, then just carry out the already-covered simplifications. If the complex fraction does contain terms, as is the case for the last two examples, then an additional step of adding fractions is necessary. For the example $\frac{\frac{3x}{5}+\frac{2}{3}}{6}$, we need to do the following: $\frac{\frac{3x}{5}+\frac{2}{3}}{6} = \frac{\frac{3}{3}\cdot\frac{3x}{5}+\frac{2}{3}\cdot\frac{5}{5}}{6} = \frac{\frac{9x}{15}+\frac{10}{15}}{6} = \frac{\frac{9x+10}{15}}{6} = \frac{\frac{9x+10}{15}}{\frac{6}{1}} = \frac{9x+10}{15}\cdot\frac{1}{6} = \frac{9x+10}{90}$. For the example $\frac{\frac{3}{5}+2}{\frac{5}{2}+\frac{22}{3x}}$, we do the following: $\frac{\frac{3}{5}+2}{\frac{5}{2}+\frac{22}{3x}} = \frac{\frac{3}{5}+2\cdot\frac{5}{5}}{\frac{3x}{3x}\cdot\frac{5}{2}+\frac{22}{3x}\cdot\frac{2}{2}} = \frac{\frac{3}{5}+\frac{10}{5}}{\frac{15x}{6x}+\frac{44}{6x}} = \frac{\frac{3+10}{5}}{\frac{15x+44}{6x}} = \frac{\frac{13}{5}}{\frac{15x+44}{6x}} = \frac{13}{5}\cdot\frac{6x}{15x+44} = \frac{78x}{5(15x+44)}$.

Finally, a manipulation that comes up occasionally in calculus has to do with multiplied fractions and the freedom to rearrange the order of the numerators or denominators. One example of this is $\frac{\sin(x)}{4\cos(x)}\cdot\frac{37}{x} = \frac{\sin(x)}{x}\cdot\frac{37}{4\cos(x)}$. The reason for changing the order of factors in the way I did has to do with the importance of grouping "$\frac{\sin(x)}{x}$" into its own fraction (which turns out to be in important expression in differential calculus).

2. Syntax options. Three important syntax issues related to fractions come up in calculus.

Firstly, there are two popular ways to write fractions, the first being "inline fractions," e.g., 2/3, and the second being "vertical fractions," e.g., $\frac{2}{3}$. I always recommend using vertical fractions, since it keeps a better visual distinction between the numerator and denominator. Furthermore, it helps in preventing a variable from morphing from being in the denominator to ending up in the numerator, e.g., 2/3*x* … 2*x*/3.

Secondly, it is crucial in calculus to rewrite a fraction that looks like "$\frac{x^3}{2}$" to appear as "$\frac{1}{2}x^3$". Although both fractions are equivalent, the second way—where we are clearly demarcating the coefficient *to the left of the variable*—is indispensable in performing some of the differentiation rules.

Lastly, be sure to use enough parentheses when using calculators. A calculator will not know that "$384 + 32.3/8$" was meant to be the computation of $(384 + 32.3)/8$. It is better to use too many pairs of parentheses than too few!