The Weekly Rigor

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Elementary Introduction to the Exponent Rules for Positive Integer Exponents
(Part 2)

Theorem 5: If $x \neq 0$, then $\frac{x^m}{x^n} = \begin{cases} x^{m-n} & \text{when } m > n \\ 1 & \text{when } m = n \\ \frac{1}{x^{n-m}} & \text{when } m < n \end{cases}$.

Proof: When $m > n$, $\frac{x^m}{x^n} = \frac{x^{m+n-n}}{x^n} = \frac{x^{n+m-n}}{x^n} \overset{\text{T1}}{\cong} \frac{x^n \cdot x^{m-n}}{x^n} = x^{m-n}$.
When $m = n$, $\frac{x^m}{x^n} = \frac{x^n}{x^n} = 1$.
When $m < n$, $\frac{x^m}{x^n} = \frac{x^m}{x^{n+m-m}} = \frac{x^m}{x^{m+n-m}} \overset{\text{T1}}{\cong} \frac{x^m}{x^m \cdot x^{n-m}} = \frac{1}{x^{n-m}}$.

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Examples: $\frac{a^8}{a^3} = a^{8-3} = a^5$. $\frac{2^3}{2^3} = 1$. $\frac{a^6}{a^{11}} = \frac{1}{a^{11-6}} = \frac{1}{a^5}$.

Theorem 6: If $y \neq 0$, then $\left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}$.

Proof: $\left(\frac{x}{y}\right)^n \overset{\text{D1}}{\cong} \overbrace{\frac{x}{y} \cdot \frac{x}{y} \cdots \frac{x}{y}}^{n\ factors} = \frac{\overbrace{x \cdot x \cdots x}^{n\ factors}}{\underbrace{y \cdot y \cdots y}_{n\ factors}} \overset{\text{D1}}{\cong} \frac{x^n}{y^n}$.

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Examples: $\left(\frac{2}{3}\right)^4 = \frac{2^4}{3^4} = \frac{16}{81}$. $\left(\frac{x}{y}\right)^5 = \frac{x^5}{y^5}$.

Theorem 7: If $y \neq 0$, then $\left(\frac{x^m}{y^n}\right)^k = \frac{x^{mk}}{y^{nk}}$.

Proof: $\left(\frac{x^m}{y^n}\right)^k = \left(\frac{(x^m)}{(y^n)}\right)^k \overset{\text{T6}}{\cong} \frac{(x^m)^k}{(y^n)^k} \overset{\text{T2}}{\cong} \frac{x^{mk}}{y^{nk}}$.

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Examples: $\left(\frac{2^5}{3^1}\right)^4 = \frac{2^{5\cdot 4}}{3^{1\cdot 4}} = \frac{2^{20}}{3^4}$. $\left(\frac{x^2}{y^3}\right)^5 = \frac{x^{2\cdot 5}}{y^{3\cdot 5}} = \frac{x^{10}}{y^{15}}$.

Theorem 8: If $y \neq 0,$ and $z \neq 0$, then $\left(\frac{w^m x^n}{y^p z^q}\right)^k = \frac{w^{mk} x^{nk}}{y^{pk} z^{qk}}$.

Proof: $\left(\frac{w^m x^n}{y^p z^q}\right)^k \overset{\text{T6}}{\cong} \frac{(w^m x^n)^k}{(y^p z^q)^k} \overset{\text{T4}}{\cong} \frac{w^{mk} x^{nk}}{y^{pk} z^{qk}}$.

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Examples: $\left(\frac{2^3 5^2}{7^3 3^9}\right)^4 = \frac{2^{3\cdot 4} 5^{2\cdot 4}}{7^{3\cdot 4} 3^{9\cdot 4}} = \frac{2^{12} 5^8}{7^{12} 3^{36}}$. $\left(\frac{w^{105} x^1}{y^5 z^2}\right)^5 = \frac{w^{105\cdot 5} x^{1\cdot 5}}{y^{5\cdot 5} z^{2\cdot 5}} = \frac{w^{525} x^5}{y^{25} z^{10}}$.

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