The Weekly Rigor

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An Essential Skill for Calculus Students: Exponents and Radicals

Exponent principles play a major role in calculus. There are three fundamental such principles: 1. Same-Base Products; 2. Power of Powers; 3. Negative Exponents.

1. Same-Base Products. Since $a^m \cdot a^n = a^{m+n}$, when multiplying numbers with the same real-number base, we add the exponents. (The exponents can be any real number.) Examples:

$$2^5 \cdot 2^3 = 2^{5+3} = 2^8. \qquad 3^{\frac{4}{5}} \cdot 3^{\frac{2}{5}} = 3^{\frac{4}{5}+\frac{2}{5}} = 3^{\frac{6}{5}}.$$

$$4^{\pi} \cdot 4^{e} = 4^{\pi+e}. \qquad \pi^{\sqrt{2}} \cdot \pi^5 = \pi^{\sqrt{2}+5}.$$

$$x^7 \cdot x^{5\pi} = x^{7+5\pi}. \qquad 5^x \cdot 5^{\sin(\pi)} = 5^{x+\sin(\pi)} = 5^{x+0} = 5^x.$$

Furthermore, since $\frac{a^m}{a^n} = a^{m-n}$, when dividing numbers with the same real-number base, we subtract the exponents. (The exponents can be any real number.)

Examples:

$$\frac{2^5}{2^3} = 2^{5-3} = 2^2. \qquad \frac{3^{\frac{4}{5}}}{3^{\frac{2}{5}}} = 3^{\frac{4}{5}-\frac{2}{5}} = 3^{\frac{2}{5}}.$$

$$\frac{4^{\pi}}{4^{e}} = 4^{\pi-e}. \qquad \frac{\pi^{\sqrt{2}}}{\pi^5} = \pi^{\sqrt{2}-5}.$$

$$\frac{x^7}{x^{5\pi}} = x^{7-5\pi}. \qquad \frac{5^x}{5^{\sin(\pi)}} = 5^{x-\sin(\pi)} = 5^{x-0} = 5^x.$$

2. Power of Powers. Since $(a^m)^n = a^{m \cdot n}$, when taking a power of a power, we multiply the exponents.

Examples:

$(2^5)^3 = 2^{5 \cdot 3} = 2^{15}$. $\left(3^{\frac{4}{5}}\right)^{\frac{2}{5}} = 3^{\frac{4}{5} \cdot \frac{2}{5}} = 3^{\frac{8}{25}}$.

$(4^\pi)^e = 4^{\pi \cdot e}$. $\left(\pi^{\sqrt{2}}\right)^5 = \pi^{5\sqrt{2}}$.

$(x^7)^{5\pi} = x^{7 \cdot 5\pi} = x^{35\pi}$. $(5^x)^{\sin(\pi)} = 5^{x \cdot \sin(\pi)} = 5^{x \cdot 0} = 5^0 = 1$.

3. Negative Exponents. Since $a^{-n} = \frac{1}{a^n}$, and $\frac{1}{a^{-n}} = a^n$, when confronted by a base taken to a negative power, we can change the fractional location of the exponential and make the exponent positive.

Examples:

$2^{-5} = \frac{1}{2^5}$. $\frac{1}{3^{-\frac{4}{5}}} = 3^{\frac{4}{5}}$.

$4^{-\pi} = \frac{1}{4^\pi}$. $\frac{1}{\pi^{-\sqrt{2}}} = \pi^{\sqrt{2}}$.

$x^{-7} = \frac{1}{x^7}$. $\frac{1}{5^{-x}} = 5^x$.

A crucial notational issue concerns the use of fractional exponents versus the use of radical signs. For example, $2^{\frac{1}{3}} = \sqrt[3]{2}$, $5^{\frac{4}{7}} = \sqrt[7]{5^4}$, and in general $a^{\frac{m}{n}} = \sqrt[n]{a^m}$. In calculus both notations are used heavily, depending on the context. When applying the derivative and integration rules, the exponential form is unavoidable. On the other hand, when engaging in algebraic manipulations to solve problems, the radical form is usually advantageous.