The Weekly Rigor

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An Essential Skill for Calculus Students: Algebraic Mal-Rules

Certain erroneous patterns crop up in algebra so often that they are given the special label “mal-rules.”

The first type of pattern might be described as “false distribution.” The first example of false distribution is where a student writes an equation such as

$$(a+b)^2 = a^2 + b^2.$$

This equation cannot be valid, since an easy counterexample can be constructed. Let $a = 1$ and $b = 1$. Now substitute these values into the left side of the pattern: $(1+1)^2 = 2^2 = 4$. Furthermore, substitute the same values into the right side of the same pattern: $1^2 + 1^2 = 1 + 1 = 2$. But $4 \neq 2$. Hence, the equation is not valid. The second example of false distribution is writing an equation of the form

$$\sqrt{a+b} = \sqrt{a} + \sqrt{b}\,.$$

Now this equation is also invalid, as shown by the following counterexample. Let $a = 16$ and $b = 9$. Substituting the values into the left side gives: $\sqrt{16+9} = \sqrt{25} = 5$. A similar substitution produces: $\sqrt{16} + \sqrt{9} = 4 + 3 = 7$. But $5 \neq 7$.

The second type of pattern might be termed “false cancellation.” This common sort of error occurs when a student writes either

$$\frac{a+b}{a} = b$$

or

$$\frac{a+b}{a} = 1 + b$$

The first “equation” can be shown to be wrong by this counterexample: Let $a = 2$ and $b = 4$. $\frac{2+4}{2} = \frac{6}{2} = 3$. But $3 \neq 4$. The second’s refutation can be shown by this counterexample: Let $a = 2$ and $b = 4$. $\frac{2+4}{2} = \frac{6}{2} = 3$. $1 + 4 = 5$. But $3 \neq 5$. Matters are not at all helped if the student writes

$$\frac{ac+b}{a} = c + b,$$

for the cancellation of the a's was illicit. (Let $a = 2$, $b = 2$, and $c = 3$.) The most common place where this latter false cancellation occurs is in the application of the quadratic formula. Many a student is tempted to do something like the following:

$$x = \frac{4 \pm \sqrt{5}}{2} = 2 \pm \sqrt{5}\,,$$

which is incorrect.

A third invalid pattern concerns a simple matter of mathematical punctuation, however young algebra students often overlook it, viz.,

$$(-a)^2 = -a^2\,.$$

Easy counterexample: Let $a = 2$. $(-2)^2 = (-2)(-2) = 4$. $-2^2 = -4$. But $4 \neq -4$.

A final mal-rule is more often a temptation than an actual commission, but it is a dangerous temptation nonetheless. This pattern may be termed "false fraction splitting," and looks like:

$$\frac{1}{a+b} = \frac{1}{a} + \frac{1}{b}.$$

A counterexample for this error is: Let $a = 2$ and $b = 4$. $\frac{1}{2+2} = \frac{1}{4}$. $\frac{1}{2} + \frac{1}{2} = 1$. But $\frac{1}{4} \neq 1$.