The Weekly Rigor

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An Essential Skill for Calculus Students: Difference Quotients
(Part 1)

The difference quotient is a key component of the formal definition of the derivative. Most calculus textbooks state the general difference quotient as

$$\frac{f(x+h)-f(x)}{h},$$

where $f(x)$ is a function. For each function, there is a unique difference quotient. For example, the difference quotient of $f(x) = x^3$ is $\frac{(x-h)^3-x^3}{h}$; the difference quotient of $f(x) = \sin(x)$ is $\frac{\sin(x+h)-\sin(x)}{h}$.

The task of a difference quotient problem is to find a way to algebraically resolve the original quotient so that the h in the original denominator can be cancelled out. There are three main types of functions that present different and challenging algebraic steps to resolve: 1. Expansion Type, e.g., $f(x) = x^2$; 2. Conjugate Type, e.g., $f(x) = \sqrt{x}$; 3. Fraction Type, e.g., $f(x) = \frac{1}{x}$. In essence, there are just three steps to solve these problems: *Write out*, *substitute*, and *resolve*.

1. Expansion Type. The simplest function of this type is $f(x) = x^2$. To resolve its difference quotient, perform the following steps:

a. Write out $f(x)$ *and* $f(x+h)$: $f(x) = x^2 \quad f(x+h) = (x+h)^2$.

b. Write out the general difference quotient and then substitute for $f(x)$ and $f(x+h)$ what you just wrote out above on the right sides of the equals sign:

$$\frac{f(x+h)-f(x)}{h} = \frac{(x+h)^2 - x^2}{h} = \cdots$$

c. Resolve the difference quotient to be able to cancel out the denominator’s *h*:

$$\cdots = \frac{x^2 + 2xh + h^2 - x^2}{h} = \frac{2xh - h^2}{h} = \frac{h(2x-h)}{h} = 2x - h\,.$$

Therefore: $\frac{f(x+h)-f(x)}{h} = 2x - h$.

2. Conjugate Type. The simplest function of this type is $f(x) = \sqrt{x}$. To resolve its difference quotient, perform the following steps:

a. Write out: $f(x) = \sqrt{x} \quad f(x+h) = \sqrt{x+h}$.

b. Substitute:

$$\frac{f(x+h)-f(x)}{h} = \frac{\sqrt{x+h}-\sqrt{x}}{h} = \cdots$$

c. Resolve:

$$\cdots = \frac{\sqrt{x+h}-\sqrt{x}}{h} \cdot \frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}} = \frac{(x+h)-x}{h\left(\sqrt{x+h}+\sqrt{x}\right)} = \frac{h}{h\left(\sqrt{x+h}+\sqrt{x}\right)} = \frac{1}{\left(\sqrt{x+h}+\sqrt{x}\right)}.$$

Therefore: $\frac{f(x+h)-f(x)}{h} = \frac{1}{\sqrt{x+h}+\sqrt{x}}$.

(Note: The h that now exists as a term inside the radical sign is okay—as you will find out early in calculus.)

3. Fraction Type. The simplest function of this type is $f(x) = \frac{1}{x}$. To resolve its difference quotient, perform the following steps:

a. Write out: $f(x) = \frac{1}{x} \quad f(x+h) = \frac{1}{x+h}$.

b. Substitute:

$$\frac{f(x+h)-f(x)}{h} = \frac{\frac{1}{x+h}-\frac{1}{x}}{h} = \cdots$$

c. Resolve:

$$\cdots = \frac{\frac{x}{x}\cdot\frac{1}{x+h}-\frac{1}{x}\cdot\frac{x+h}{x+h}}{h} = \frac{\frac{x}{x(x+h)}-\frac{(x+h)}{x(x+h)}}{h} = \frac{\frac{x-(x+h)}{x(x+h)}}{h} = \frac{\frac{x-x-h}{x(x+h)}}{h} = \frac{\frac{-h}{x(x+h)}}{h} =$$

$$= \frac{\frac{-h}{x(x+h)}}{\frac{h}{1}} = \frac{-h}{x(x+h)} \cdot \frac{1}{h} = \frac{-1}{x(x+h)}.$$

Therefore, $\frac{f(x+h)-f(x)}{h} = \frac{-1}{x(x+h)}$.