The Weekly Rigor

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An Essential Skill for Calculus Students: Difference Quotients

(Part 2)

A challenging combination of the aforementioned types is a mixture of types 2 and 3, e.g., the function $f(x)=\frac{1}{\sqrt{x}}$. To resolve its difference quotient, perform the following steps:

a. Write out: $f(x)=\frac{1}{\sqrt{x}}$ $\quad f(x+h)=\frac{1}{\sqrt{x+h}}$.

b. Substitute:

$$\frac{f(x+h)-f(x)}{h}=\frac{\frac{1}{\sqrt{x+h}}-\frac{1}{\sqrt{x}}}{h}=\cdots$$

c. Resolve:

$$\cdots=\frac{\frac{\sqrt{x}}{\sqrt{x}}\cdot\frac{1}{\sqrt{x+h}}-\frac{1}{\sqrt{x}}\cdot\frac{\sqrt{x+h}}{\sqrt{x+h}}}{h}=\frac{\frac{\sqrt{x}}{\sqrt{x}\sqrt{x+h}}-\frac{\sqrt{x+h}}{\sqrt{x}\sqrt{x+h}}}{h}=\frac{\frac{\sqrt{x}-\sqrt{x+h}}{\sqrt{x}\sqrt{x+h}}}{h}=$$

$$=\frac{\frac{\sqrt{x}-\sqrt{x+h}}{\sqrt{x}\sqrt{x+h}}\cdot\frac{\sqrt{x}+\sqrt{x+h}}{\sqrt{x}+\sqrt{x+h}}}{h}=\frac{\frac{x-(x+h)}{\sqrt{x}\sqrt{x+h}\left(\sqrt{x}+\sqrt{x+h}\right)}}{h}=\frac{\frac{x-x-h}{\sqrt{x}\sqrt{x+h}\left(\sqrt{x}+\sqrt{x+h}\right)}}{h}=$$

$$=\frac{\frac{-h}{\sqrt{x}\sqrt{x+h}\left(\sqrt{x}+\sqrt{x+h}\right)}}{h}=\frac{\frac{-h}{\sqrt{x}\sqrt{x+h}\left(\sqrt{x}+\sqrt{x+h}\right)}}{\frac{h}{1}}=\frac{-h}{\sqrt{x}\sqrt{x+h}\left(\sqrt{x}+\sqrt{x+h}\right)}\cdot\frac{1}{h}=$$

$$=\frac{-1}{\sqrt{x}\sqrt{x+h}\left(\sqrt{x}+\sqrt{x+h}\right)}.$$

Therefore: $\frac{f(x+h)-f(x)}{h}=\frac{-1}{\sqrt{x}\sqrt{x+h}\left(\sqrt{x}+\sqrt{x+h}\right)}$.

A final challenging variation looks similar to a Conjugate Type. It is $f(x) = \sqrt[3]{x}$. However, to resolve it requires an appeal to the difference of cubes formula from algebra,

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2).$$

Here is a procedure for resolving this function's difference quotient, where $a = \sqrt[3]{x + h}$ and $b = \sqrt[3]{x}$.

a. Write out: $f(x) = \sqrt[3]{x} \quad f(x + h) = \sqrt[3]{x + h}$.

b. Substitute:

$$\frac{f(x + h) - f(x)}{h} = \frac{\sqrt[3]{x + h} - \sqrt[3]{x}}{h} = \cdots$$

c. Resolve:

$$\cdots = \frac{\sqrt[3]{x + h} - \sqrt[3]{x}}{h} \cdot \frac{\sqrt[3]{(x + h)^2} + \sqrt[3]{x + h}\sqrt[3]{x} + \sqrt[3]{x^2}}{\sqrt[3]{(x + h)^2} + \sqrt[3]{x + h}\sqrt[3]{x} + \sqrt[3]{x^2}} =$$

$$= \frac{\sqrt[3]{(x + h)^3} + \sqrt[3]{(x + h)^2}\sqrt[3]{x} + \sqrt[3]{x + h}\sqrt[3]{x^2} - \sqrt[3]{(x + h)^2}\sqrt[3]{x} - \sqrt[3]{x + h}\sqrt[3]{x^2} - \sqrt[3]{x^3}}{h\left(\sqrt[3]{(x + h)^2} + \sqrt[3]{x + h}\sqrt[3]{x} + \sqrt[3]{x^2}\right)} =$$

$$= \frac{(x + h) + \sqrt[3]{(x + h)^2}\sqrt[3]{x} - \sqrt[3]{(x + h)^2}\sqrt[3]{x} + \sqrt[3]{x + h}\sqrt[3]{x^2} - \sqrt[3]{x + h}\sqrt[3]{x^2} - x}{h\left(\sqrt[3]{(x + h)^2} + \sqrt[3]{x + h}\sqrt[3]{x} + \sqrt[3]{x^2}\right)} =$$

$$= \frac{x + h - x}{h\left(\sqrt[3]{(x + h)^2} + \sqrt[3]{x + h}\sqrt[3]{x} + \sqrt[3]{x^2}\right)} = \frac{h}{h\left(\sqrt[3]{(x + h)^2} + \sqrt[3]{x + h}\sqrt[3]{x} + \sqrt[3]{x^2}\right)} =$$

$$= \frac{1}{\sqrt[3]{(x + h)^2} + \sqrt[3]{x + h}\sqrt[3]{x} + \sqrt[3]{x^2}} .$$

Therefore: $\frac{f(x+h)-f(x)}{h} = \frac{1}{\sqrt[3]{(x+h)^2}+\sqrt[3]{x+h}\sqrt[3]{x}+\sqrt[3]{x^2}}$.