# The Weekly Rigor

No. 91 “A mathematician is a machine for turning coffee into theorems.” March 19, 2016


## An Essential Skill for Calculus Students: Functions

Functions are a central issue in calculus. Three issues about functions that need attention are: 1. the general definition of the function concept; 2. the domain and range of a function; 3. the understanding of function notation.

1. The general definition of the function concept. A simple way to understand what functions do is to imagine a “rule machine” that takes in inputs and produces outputs. Let $x$ represent the inputs and $y$ the outputs. Now this machine produces the outputs according to a rule specified by an equation. For example, one rule machine could be:

$$x \Longrightarrow \boxed{y = 2x + 3} \Longrightarrow y$$

In words, we could say that the rule machine takes an input, multiplies it by 2, and then adds 3 to the product. Some examples of inputs becoming outputs would be:

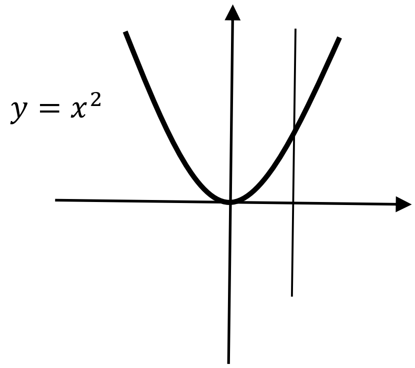
$$1 \Longrightarrow \boxed{y = 2(1) + 3} \Longrightarrow 5$$

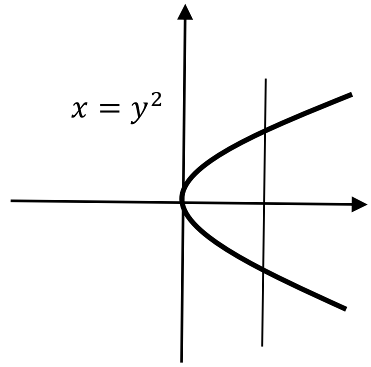
$$2 \Longrightarrow \boxed{y = 2(2) + 3} \Longrightarrow 7$$

$$3 \Longrightarrow \boxed{y = 2(3) + 3} \Longrightarrow 9$$

Typically the inputs and outputs are communicated as *ordered pairs* of the form $(x, y)$. Hence, for the examples about we have $(1, 5)$, $(2, 7)$, and $(3, 9)$.

The only restriction on the function rule machine is that each input produces a *unique* output. When graphing functions, this restriction is often described as the “vertical line test.” For example, the equation $y = x^2$ is a function, since it passes the vertical line test. (See the left graph below.) However, the equation $x = y^2$ is not a function, since it fails the same test. (See the right graph below.)




A last issue regarding the basic concept of functions concerns the letters standing for the inputs and outputs. Let us adopt the shorthand $x \to y$ to stand in place of the rule machine diagrams we used above. With that shorthand, the student will often see $t \to P$, where *t* (for time) is the input and $P$ is the output. Other popular choices are $\theta \to y$ (often used in trigonometric applications) and $t \to x$.

2. The domain and range of a function. In addition to understanding the basic function concept, the calculus student will face determining the domain and range of a given function. In terms of the rule machine metaphor, the domain of a function is just the set of all the inputs; the range is just the set of all the outputs. Nothing more, nothing less.

Often the domain and range will have to be determined from the function's graph. Here it is important to grasp the principle that the domain is found by tracking the graph relative to the *x*-axis going left to right. Whatever *x* values are included in the graph constitute the domain of the function. In a similar manner, the range is found by tracking the graph relative to the *y*-axis going from the bottom to the top. Whatever *y* values are included in the graph constitute the range of the function.

3. The understanding of function notation. The ordered pair $(x, y)$ can be written using function notation, which consists of an equation such as $f(x) = y$. Using this equation, we can identify the three parts to this notation. From left to right, the first part is the name of the function. It can be a single letter or a whole word. The second part is the pair of parentheses, which contain the input(s). Each input is called an "argument" of the function. Finally, the third part—on the right side of the equality sign—is the output corresponding to the given input. Each such output is called a "value" of the function.

$$f(x) = y$$

name argument value

Returning to the three ordered pairs used in the rule machine, we would have $f(1) = 5$, $f(2) = 7$, and $f(3) = 9$.