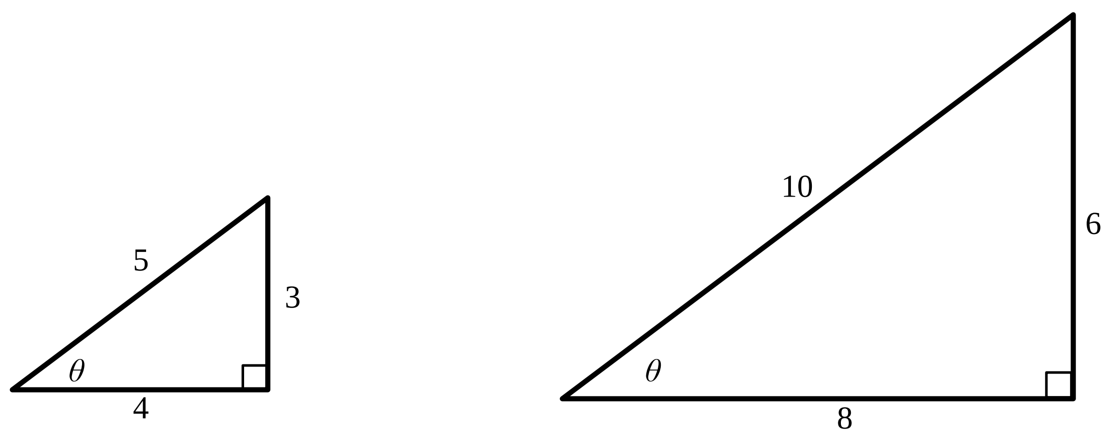
The Weekly Rigor

No. 97 "A mathematician is a machine for turning coffee into theorems." April 30, 2016

An Essential Skill for Calculus Students: Trigonometry
(Part 1)

The needed trigonometric knowledge a successful calculus student must have falls into five categories: 1. The basis of the trigonometric functions on similar right triangles; 2. The domain versus range of trigonometric functions with respect to similar right triangles and the unit circle; 3. Needed trigonometric identities; 4. Radian measure; 5. Unit circle measurements.

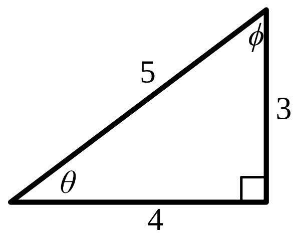
1. Similar right triangles. Consider two similar right triangles.



The angle θ is the same for both triangles. In the left triangle, the leg opposite to θ has length 3 and the leg adjacent to θ has length 4. Hence, the ratio of opposite over adjacent equals $\frac{3}{4}$. Notice that in the case of the triangle on the right the ratio of opposite leg to θ over the adjacent leg to θ equals $\frac{6}{8}=\frac{3}{4}$. Pick any other similar triangles to these first two and the same ratio of $\frac{3}{4}$ appears for the leg opposite to θ divided by leg adjacent to θ. For brevity, let us abbreviate the ratio as $\frac{\text{OPP}}{\text{ADJ}}$. The ratio $\frac{\text{OPP}}{\text{ADJ}}$ is called "tangent." That is, the mathematical tangent of $\theta = \frac{\text{OPP}}{\text{ADJ}}$, with the shorthand $\tan(\theta) = \frac{\text{OPP}}{\text{ADJ}}$. Note carefully that the tangent of angle θ *is* the ratio of the opposite leg divided by the adjacent leg.

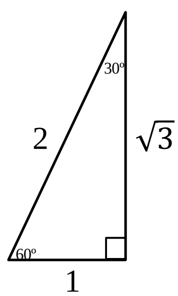
Furthermore, in all the above triangles, the ratio of the leg opposite θ divided by the hypotenuse equals $\frac{3}{5}$, viz., $\frac{\text{OPP}}{\text{HYP}} = \frac{3}{5}$. The trigonometric function that specifies the ratio $\frac{\text{OPP}}{\text{HYP}}$ relative to θ is called "sine." More briefly, $\sin(\theta) = \frac{\text{OPP}}{\text{HYP}}$. Lastly, "cosine" is defined as $\cos(\theta) = \frac{\text{ADJ}}{\text{HYP}}$.

It is crucial to remember that sine, cosine, and tangent are defined *relative to an angle*. For example, in the following triangle

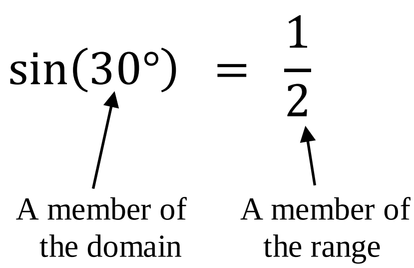


$\sin(\theta) = \frac{3}{5}$, but $\sin(\phi) = \frac{4}{5}$; $\tan(\theta) = \frac{3}{4}$, but $\tan(\phi) = \frac{4}{3}$. Note however that $\cos(\theta) = \frac{4}{5} = \sin(\phi)$. That is, the *co*sine of θ is equal to the sine of ϕ. But $\theta + \phi = 90°$, that is, θ and ϕ are *complementary* angles. Hence, "cosine" is short for "complementary angle's sine."

2. Domain and range. When dealing with right triangles, the domain of the sine, cosine, and tangent functions consists of *acute angles*. The range of the same functions consists of ratios of side lengths. For example, in the following triangle:



we have



When the trigonometric functions are applied to the unit circle, the domain is expanded to include all real numbers and the range is the interval $[-1, 1]$.