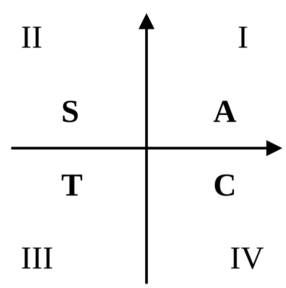
The Weekly Rigor

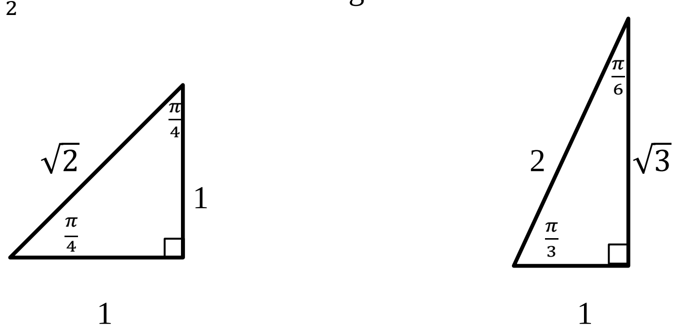
No. 100 “A mathematician is a machine for turning coffee into theorems.” May 21, 2016

An Essential Skill for Calculus Students: Trigonometry
(Part 4)

Now $\sin\left(\frac{11\pi}{6}\right) = -\frac{1}{2}$ and $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$; $\cos\left(\frac{11\pi}{6}\right) = \frac{\sqrt{3}}{2}$ and $\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$, etc. In other words, there is a close connection between the trigonometric values of an angle in standard position and its related reference angles. The only difference between the two angles involves occasional opposition of signs: Reference angles always produce positive values, but angles in standard position sometimes produce negative values. The quick way to deal with that issue is the mnemonic device “**A**ll **S**tudents **T**ake **C**alculus”—shorthand for: All trigonometric functions are positive in Quadrant I; Sine is positive in Quadrant II; Tangent is positive in Quadrant III; and Cosine is positive in Quadrant IV.



Now we can address the question of how to compute the value of the reference angles, e.g., $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$. The two reference triangles come to our aid:



(Cf. *WR* no. 95 for the proper construction of these triangles.) We just read off from them the needed values. For example, for computing $\sin\left(\frac{\pi}{6}\right)$, look at the triangle that contains the angle with measure $\frac{\pi}{6}$ and apply SOHCAHTOA: $\sin\left(\frac{\pi}{6}\right) = \frac{\text{OPP}}{\text{HYP}} = \frac{1}{2}$.

To put all of this together, note the following procedure in the example where we compute the value of $\csc\left(\frac{5\pi}{3}\right)$.

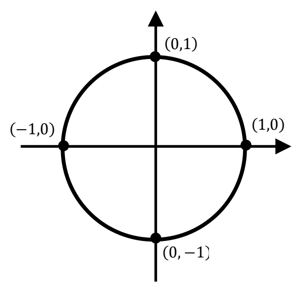
1. $\csc\left(\frac{5\pi}{3}\right) = \frac{1}{\sin\left(\frac{5\pi}{3}\right)}$ and the reference angle for $\frac{5\pi}{3}$ is $\frac{\pi}{3}$.

2. Consulting the 30º-60º-90º triangle, we find that $\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$. Hence, $\csc\left(\frac{\pi}{3}\right) = \frac{2}{\sqrt{3}}$.

3. But $\frac{5\pi}{3}$ is in Quadrant IV, and sine and cosecant are both negative there (ASTC).

4. Therefore, $\csc\left(\frac{5\pi}{3}\right) = -\frac{2}{\sqrt{3}}$.

This procedure obviates the need to consult the unit circle except for the special four points $(1,0)$, $(0,1)$, $(-1,0)$, $(0,-1)$, which correspond to the angles in standard position that have a measurement equal to a multiple of 90º.



However, for any x-value on the unit circle, $x = \cos(\theta)$ and for any y-value $y = \sin(\theta)$, where θ is an angle in standard position. Hence, the values for cosine and sine corresponding to the special four points can be computed easily.

$$(1,0) \implies \theta = 0 \implies \cos(0) = 1 \quad \sin(0) = 0$$

$$(0,1) \implies \theta = \frac{\pi}{2} \implies \cos\left(\frac{\pi}{2}\right) = 0 \quad \sin\left(\frac{\pi}{2}\right) = 1$$

$$(-1,0) \implies \theta = \pi \implies \cos(\pi) = -1 \quad \sin(\pi) = 0$$

$$(0,-1) \implies \theta = \frac{3\pi}{2} \implies \cos\left(\frac{3\pi}{2}\right) = 0 \quad \sin\left(\frac{3\pi}{2}\right) = -1$$