The Weekly Rigor

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The Sum of the Cubes of Any Three Consecutive Integers is Divisible by 9

INTRODUCTION

We prove the claim in this article by means of some basic principles of modular arithmetic. We use modular arithmetic every time we use a 12-hour clock, where the numbers “wrap around” upon reaching 12–that number being the “modulus.” Below we will see a way to use modulo 9 to quickly prove that, for example, $44^3 + 45^3 + 46^3$ is divisible by 9.

Definition 1:

$$a|b$$

if and only if $a \neq 0$ and there exists an integer *c* such that

$$ac = b.$$

For “$a|b$,” *b* is said to be divisible by *a*, *b* is a multiple of *a*, *a* divides *b*, *a* goes into *b*, or *a* is a factor of *b*.

If $a \neq 0$ and *b* is not divisible by *a*, then we write

$$a \nmid b.$$

Examples: $2|6$, $4 \nmid 6$, $3 \nmid 4$, $2|-4$, $5|0$, $1|0$, $3|3$.

Definition 2: We say that *a* and *b* are *congruent modulo* n if and only if

$$n|(a - b).$$

We denote this relationship as

$$a \equiv b \pmod{n},$$

and read these symbols as “*a* is congruent to *b* modulo *n*.”

Examples: $9 \equiv 3 \pmod{2}$, since $2|6$ and hence $2|(9-3)$. $0 \equiv -4 \pmod{2}$, since $2|-4$ and hence $2|(0-4)$. $4 \equiv 4 \pmod{5}$, since $5|0$ and hence $5|(4-4)$.

Theorem 1: $9^3 \equiv 0^3 \pmod 9$

Proof: $9 \cdot 9^2 = 9^3$. Hence, $9|9^3$, i.e., $9|(9^3 - 0)$, by Definition 1. Therefore, $9^3 \equiv 0 \equiv 0^3 \pmod 9$, by Definition 2.

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Theorem 2: $10 \equiv 1^3 \pmod 9$

Proof: $9 \cdot 111 = 999$. Hence, $9|999$, i.e., $9|(1000 - 1)$, by Definition 1. So, $9|(10^3 - 1)$. Therefore, $10^3 \equiv 1 \equiv 1^3 \pmod 9$, by Definition 2.

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Theorem 3: The sum of the cubes of any three consecutive integers is divisible by 9.

Preliminary Remark: This generalization can be suggested by examples such as the following: $(-1)^3 + 0^3 + 1^3 = 0$, $0^3 + 1^3 + 2^3 = 9$, $1^3 + 2^3 + 3^3 = 36$,…

Proof: Let $n, n+1, n+2$ be three consecutive integers. $n^3 + (n+1)^3 + (n+2)^3$ is divisible by 9 if and only if $9|[n^3 + (n+1)^3 + (n+2)^3]$, by Definition 1. But $9|\{[n^3 + (n+1)^3 + (n+2)^3] - 0\}$ if and only if $n^3 + (n+1)^3 + (n+2)^3 \equiv 0 \pmod 9$, for $n = 0, 1, \ldots, 8$ by Definition 2. Now,

$$0^3 + 1^3 + 2^3 \equiv 9 \equiv 0 \pmod 9$$
$$1^3 + 2^3 + 3^3 \equiv 36 \equiv 0 \pmod 9$$
$$2^3 + 3^3 + 4^3 \equiv 99 \equiv 0 \pmod 9$$
$$3^3 + 4^3 + 5^3 \equiv 216 \equiv 0 \pmod 9$$
$$4^3 + 5^3 + 6^3 \equiv 405 \equiv 0 \pmod 9$$
$$5^3 + 6^3 + 7^3 \equiv 684 \equiv 0 \pmod 9$$
$$6^3 + 7^3 + 8^3 \equiv 1071 \equiv 0 \pmod 9$$
$$7^3 + 8^3 + 9^3 \overset{\text{T1}}{\equiv} 7^3 + 8^3 + 0^3 \equiv 855 \equiv 0 \pmod 9$$
$$8^3 + 9^3 + 10^3 \overset{\text{T1}}{\equiv} 8^3 + 0^3 + 10^3 \overset{\text{T2}}{\equiv} 8^3 + 0^3 + 1^3 \equiv 513 \equiv 0 \pmod 9$$

Therefore, $n^3 + (n+1)^3 + (n+2)^3$ is divisible by 9.

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