The Weekly Rigor

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SAT Math Test Problem Children: Complex Numbers

(Part 1)

INTRODUCTION

The College Board has posted 280 math problems consistent with the new version of the SAT, which was launched earlier this year. These problems show up on four practice exams for the SAT and one practice exam for the PSAT. Certain categories of math questions come up repeatedly in the practice exams and are likely to challenge even the best of math students. I call these categories “problem children.” This article will address the category dealing with complex numbers.

In essence, for the problems involving complex numbers, there are two definitions and three operations to know about.

Firstly, like the Greek letter π, which has a special value in mathematics, the letter i also has a particular value. That is, $i^2 = 1$ or $i = \sqrt{-1}$. Secondly, a “complex number” is a number of the form $a + bi$, where both a and b are real numbers, and $i = \sqrt{-1}$. In the expression $a + bi$, the number a is called the “real part” and the number bi is called the “imaginary part.” For example, in the complex number

$$3 + 4i,$$

the number 3 is the real part and $4i$ is the imaginary part.

Secondly, the three operations that come up are addition/subtraction, multiplication, and division.

For addition and subtraction, we add (subtract) the real parts and then add (subtract) the imaginary parts. For example,

$$\begin{aligned}(3 + 5i) + (4 - i) &= 3 + 4 + 5i - i \\ &= (3 + 4) + (5 - 1)i \\ &= 7 + 4i\end{aligned}$$

The easiest way to handle multiplication is by the FOIL method. Remember to apply the definition that $i^2 = 1$. For example,

$$
\begin{aligned}
(3+5i)(4-i) &= \overbrace{3\cdot 4}^{\text{F}} - \overbrace{3\cdot i}^{\text{O}} + \overbrace{4\cdot 5i}^{\text{I}} - \overbrace{5i^2}^{\text{L}} \\
&= 12 - 3i + 20i - 5i^2 \\
&= 12 + 17i - 5(-1) \\
&= 12 + 17i + 5 \\
&= 12 + 5 + 17i \\
&= 17 + 17i
\end{aligned}
$$

Finally we come to the division of two complex numbers, for example $\frac{3+5i}{4-i}$. Crucial to solving such problems is the concept of the "conjugate" of a complex number. The conjugate of $4 - i$ is $4 + i$. Likewise, the conjugate of $4 + i$ is $4 - i$. In general, $a + bi$ and $a - bi$ are conjugates of one another. Now to solve $\frac{3+5i}{4-i}$, proceed as follows:

$$
\begin{aligned}
\frac{3+5i}{4-i} &= \frac{3+5i}{4-i}\cdot\frac{4+i}{4+i} \\
&= \frac{(3+5i)}{(4-i)}\cdot\frac{(4+i)}{(4+i)} \\
&= \frac{3\cdot 4 + 3i + 4\cdot 5i + 5i^2}{4\cdot 4 + 4i - 4i - i^2} \\
&= \frac{12 + 3i + 20i + 5i^2}{16 - i^2} \\
&= \frac{12 + 23i + 5i^2}{16 - i^2} \\
&= \frac{12 + 23i + 5(-1)}{16 - (-1)} \\
&= \frac{12 + 23i - 5}{16 + 1} \\
&= \frac{7 + 23i}{17} \\
&= \frac{7}{17} + \frac{23}{17}i
\end{aligned}
$$