The Weekly Rigor

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Three Main Properties of Set Inclusion

Property 1: $A \subseteq A$ for every A.
Property 2: $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.
Property 3: If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

Definition 1: $A \subseteq B$ if and only if $(\forall x)(x \in A \longrightarrow x \in B)$.

Definition 2: $A = B$ if and only if $(\forall x)(x \in A \longleftrightarrow x \in B)$.

Theorem 1 (Property 1): $A \subseteq A$ for every A.

Proof: $(\forall x)(x \in A \longrightarrow x \in A)$. Hence, $A \subseteq A$, by Definition 1.

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Theorem 2: If $A = B$, then $A \subseteq B$ and $B \subseteq A$.

Proof: Suppose that $A = B$. $A \subseteq A$, by Theorem 1. Hence, $A \subseteq B$ and $B \subseteq A$, by substitutions of “B” for “A.”

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Theorem 3: If $A \subseteq B$ and $B \subseteq A$, then $A = B$.

Proof: Suppose that $A \subseteq B$ and $B \subseteq A$. Hence, $(\forall x)(x \in A \longrightarrow x \in B)$ and $(\forall x)(x \in B \longrightarrow x \in A)$, by Definition 1. So, $(\forall x)(x \in A \longleftrightarrow x \in B)$. Thus, $A = B$, by Definition 2.

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Theorem 4 (Property 2): $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.

Proof: By Theorems 2 and 3.

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Theorem 5 (Property 3): If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

Proof: Suppose that $A \subseteq B$ and $B \subseteq C$. Hence, $(\forall x)(x \in A \longrightarrow x \in B)$ and $(\forall x)(x \in B \longrightarrow x \in C)$, by Definition 1. So, $(\forall x)(x \in A \longrightarrow x \in C)$. Therefore, $A \subseteq C$, by Definition 1.

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