

The Weekly Rigor

No. 155

“A mathematician is a machine for turning coffee into theorems.”

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Seven Essential Properties of Absolute Value (Part 1)

INTRODUCTION

The seven essential properties of the absolute value function are the following:

Property 1: For every real number x , $|x| \geq 0$.

Property 2: If x is any real number and a is a positive real number, then

$$|x| < a \text{ if and only if } -a < x < a.$$

Property 3: If x and b are any real numbers and a is positive, then

$$|x - b| < a \text{ if and only if } b - a < x < b + a.$$

Property 4: If x is any real number and a is a positive real number, then

$$|x| > a \text{ if and only if either } x > a \text{ or } x < -a.$$

Property 5: If a and b are any real numbers, then $|a - b| = |b - a|$.

Property 6: If a and b are any real numbers, then $|a| \cdot |b| = |a \cdot b|$.

Property 7 (The Triangle Inequality): If a and b are any real numbers, then

$$|a + b| \leq |a| + |b|.$$

Examples of Property 1 include the following:

$$|3| = 3 \quad |-3| = 3 \quad |0| = 0$$

Examples of Property 2 include the following:

$$|3| < 4 \text{ if and only if } -4 < 3 < 4$$

$$|-3| < 4 \text{ if and only if } -4 < -3 < 4$$

$$|0| < 4 \text{ if and only if } -4 < 0 < 4$$

Examples of Property 3 include the following:

$$|5 - 2| < 4 \text{ if and only if } 2 - 4 < 5 < 2 + 4$$

$$|-3 - (-1)| < 4 \text{ if and only if } -1 - 4 < -3 < -1 + 4$$

Examples of Property 4 include the following:

$$|4| > 3 \quad \text{if and only if} \quad \text{either } 4 > 3 \text{ or } 4 < -3$$

$$|-4| > 3 \quad \text{if and only if} \quad \text{either } -4 > 3 \text{ or } -4 < -3$$

Examples of Property 5, sometimes said to be the “distance” between the numbers a and b , include the following:

$$|4 - 3| = |3 - 4|$$

$$|0 - 2| = |2 - 0|$$

$$|-4 - 3| = |2 - (-3)| = |2 + 3|$$

$$|-4 - (-3)| = |-4 + 3| = |(-3) - (-4)| = |-3 + 4|$$

Examples of Property 6 include the following:

$$|3| \cdot |2| = |3 \cdot 4|$$

$$|0| \cdot |-3| = |0 \cdot (-3)|$$

$$|-3| \cdot |-2| = |(-3) \cdot (-4)|$$

Examples of Property 7, the “Triangle Inequality,” include the following:

$$|3 + 4| \leq |3| + |4|$$

$$|0 + 4| \leq |0| + |4|$$

$$|-3 + 1| \leq |-3| + |1|$$

$$|-3 + (-1)| \leq |-3| + |-1|$$

And now to the proofs.

“Only he who never plays, never loses.”