The Weekly Rigor

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Seven Essential Properties of Absolute Value
(Part 2)

Definition 1: The *absolute value* of a real number x, denoted by "$|x|$," is defined in the following manner:

$$|x| = \begin{cases} x \ \ if \ x \geq 0 \\ -x \ if \ x < 0 \end{cases}$$

Theorem 1: For every real number x, $|x| \geq 0$.

Proof: Either $x \geq 0$ or $x < 0$.

Case 1: Suppose that $x \geq 0$. Hence, $|x| \stackrel{\text{D1}}{=} x$. So, $|x| \geq 0$, by substitution.

Case 2: Suppose that $x < 0$. Hence, $|x| \stackrel{\text{D1}}{=} -x$ and $-x > 0$. So, $|x| > 0$, by substitution. Thus, $|x| \geq 0$.

In either case, $|x| \geq 0$.

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Theorem 2: If x is any real number and a is a positive real number, then
$|x| < a$ if and only if $-a < x < a$.

Proof: Suppose that x is any real number and a is a positive real number.
Suppose that $|x| < a$. Either $x \geq 0$ or $x < 0$.

Case 1: Suppose that $x \geq 0$. Hence, $|x| \stackrel{\text{D1}}{=} x$. So, $x < a$, by substitution. Thus, $0 \leq x < a$. Hence, $-a < 0$. Accordingly, $-a < x < a$.

Case 2: Suppose that $x < 0$. Hence, $|x| \stackrel{\text{D1}}{=} -x$. So, $-x < a$, by substitution. Thus, $-a < x$. Hence, $-a < x < 0$. So, $0 < a$. Thus, $x < a$. Accordingly, $-a < x < a$.

In either case, $-a < x < a$. Consequently, if $|x| < a$, then $-a < x < a$.

Suppose that $-a < x < a$. Either $x \geq 0$ or $x < 0$.

Case 1: Suppose that $x \geq 0$. Hence, $|x| \overset{\text{D1}}{=} x$. So, $|x| < a$, by substitution.

Case 2: Suppose that $x < 0$. Hence, $|x| \overset{\text{D1}}{=} -x$. But since $-a < x$, $-x < a$. So, $|x| < a$, by substitution.

In either case, $|x| < a$. Consequently, if $-a < x < a$, then $|x| < a$.

Therefore, $|x| < a$ if and only if $-a < x < a$.

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Theorem 3: If x and b are any real numbers and a is positive, then

$$|x - b| < a \text{ if and only if } b - a < x < b + a.$$

Proof: Suppose that x and b are any real numbers and that a is positive. Hence, $x - b$ is a real number. So,

$$|x - b| < a \text{ if and only if } -a < x - b < a,$$

by Theorem 2. Therefore,

$$|x - b| < a \text{ if and only if } b - a < x < b + a.$$

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