The Weekly Rigor

No. 157 "A mathematician is a machine for turning coffee into theorems." June 24, 2017

Seven Essential Properties of Absolute Value

(Part 3)

Theorem 4: If x is any real number and a is a positive real number, then

$$|x| > a \text{ if and only if either } x > a \text{ or } x < -a.$$

Proof: Suppose that x is any real number and a is a positive real number.

Suppose that $|x| > a$. Either $x \geq 0$ or $x < 0$.

<u>Case 1:</u> Suppose that $x \geq 0$. Hence, $|x| \overset{\text{D1}}{=} x$. So, $x > a$, by substitution. Thus, either $x > a$ or $x < -a$.

<u>Case 2:</u> Suppose that $x < 0$. Hence, $|x| \overset{\text{D1}}{=} -x$. So, $-x > a$, by substitution. Thus, $x < -a$. Hence, either $x > a$ or $x < -a$.

In either case, either $x > a$ or $x < -a$. Consequently, if $|x| > a$, then either $x > a$ or $x < -a$..

Suppose that either $x > a$ or $x < -a$. Either $x \geq 0$ or $x < 0$.

<u>Case 1:</u> Suppose that $x > a$. Hence, $x > 0$, since $a > 0$. So, $|x| \overset{\text{D1}}{=} x$. Thus, $|x| > a$, by substitution.

<u>Case 2:</u> Suppose that $x < -a$. Hence, $-x > a$. So, $-x > 0$, since $a > 0$. Thus, $x < 0$. Hence, $|x| \overset{\text{D1}}{=} -x$. So, $|x| > a$, by substitution.

In either case, $|x| > a$. Consequently, if either $x > a$ or $x < -a$, then $|x| > a$.

Therefore, $|x| > a$ if and only if either $x > a$ or $x < -a$.

■

Theorem 5: If a and b are any real numbers, then $|a-b| = |b-a|$.

Proof: Suppose that a and b are any real numbers. Either $a-b \geq 0$ or $a-b<0$.

Case 1: Suppose that $a-b \geq 0$. Hence, $-(b-a) \geq 0$. So
$|a-b| \overset{\text{D1}}{\cong} a-b = -(b-a) \overset{\text{D1}}{\cong} |b-a|$.

Case 2: Suppose that $a-b<0$. Hence, $b-a>0$. So,
$|a-b| \overset{\text{D1}}{\cong} -(a-b) = b-a \overset{\text{D1}}{\cong} |b-a|$.

In either case, $|a-b| = |b-a|$.

Therefore, If a and b are any real numbers, then $|a-b| = |b-a|$.

■