The Weekly Rigor

No. 158 "A mathematician is a machine for turning coffee into theorems." July 1, 2017

Seven Essential Properties of Absolute Value
(Part 4)

Theorem 6: If a and b are any real numbers, then $|a| \cdot |b| = |a \cdot b|$.

Proof: Suppose that a and b are any real numbers. Exactly one of the four following possibilities holds: i. $a \geq 0$ and $b \geq 0$; ii. $a < 0$ and $b < 0$; iii. $a \geq 0$ and $b < 0$; iv. $a < 0$ and $b \geq 0$.

Case 1: Suppose that $a \geq 0$ and $b \geq 0$. Hence, $|a| \overset{D1}{=} a$, $|b| \overset{D1}{=} b$, and $ab \geq 0$. So, $|a| \cdot |b| = ab \overset{D1}{=} |a \cdot b|$.

Case 2: Suppose that $a < 0$ and $b < 0$. Hence, $|a| \overset{D1}{=} -a$, $|b| \overset{D1}{=} -b$, and $ab > 0$. So, $ab \geq 0$. Thus, $|a| \cdot |b| = (-a)(-b) = ab \overset{D1}{=} |a \cdot b|$.

Case 3: Suppose that $a \geq 0$ and $b < 0$. Hence, either $a = 0$ or $a > 0$.

Case 3a: Suppose that $a = 0$. Hence, $|a| \overset{D1}{=} a$, $|b| \overset{D1}{=} -b$, and $ab = 0$. So, $|a| \cdot |b| = a(-b) = 0(-b) = 0 = ab \overset{D1}{=} |a \cdot b|$.

Case 3b: Suppose that $a > 0$. Hence, $|a| \overset{D1}{=} a$, $|b| \overset{D1}{=} -b$, and $ab < 0$. So, $|a| \cdot |b| = a(-b) = -(ab) \overset{D1}{=} |a \cdot b|$.

In either case, $|a| \cdot |b| = |a \cdot b|$.

Case 4: Suppose that $a < 0$ and $b \geq 0$. Hence, either $b = 0$ or $b > 0$.

Case 4a: Suppose that $b = 0$. Hence, $|a| \overset{D1}{=} -a$, $|b| \overset{D1}{=} b$, and $ab = 0$. So, $|a| \cdot |b| = (-a)b = (-a)0 = 0 = ab \overset{D1}{=} |a \cdot b|$.

Case 4b: Suppose that $b > 0$. Hence, $|a| \overset{D1}{=} -a$, $|b| \overset{D1}{=} b$, and $ab < 0$. So, $|a| \cdot |b| = (-a)b = -(ab) \overset{D1}{=} |a \cdot b|$.

In either case, $|a| \cdot |b| = |a \cdot b|$.

In all four cases, $|a| \cdot |b| = |a \cdot b|$.

Therefore, If a and b are any real numbers, then $|a| \cdot |b| = |a \cdot b|$.

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Theorem 7: For every real number x,

$$-|x| \leq x \leq |x|.$$

Proof: Suppose that x is a real number. Hence, either $x \geq 0$ or $x < 0$.

Case 1: Suppose that $x \geq 0$. Hence, $-x \leq 0$ and $|x| \overset{\text{D1}}{=} x$. So, $-x \leq 0 \leq x$. Thus, $-x \leq x \leq x$. Hence, $-|x| \leq x \leq |x|$, by substitution.

Case 2: Suppose that $x < 0$. Hence, $-x > 0$ and $|x| \overset{\text{D1}}{=} -x$. So, $x < 0 < -x$. Thus, $x \leq x \leq -x$. Hence, since $-|x| = x$, $-|x| \leq x \leq |x|$, by substitution.

In either case, $-|x| \leq x \leq |x|$.

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Theorem 8 (The Triangle Inequality): If a and b are any real numbers, then

$$|a + b| \leq |a| + |b|.$$

Proof: Suppose that a and b are any real numbers. Hence, $-|a| \leq a \leq |a|$ and $-|b| \leq b \leq |b|$, by Theorem 7. So, $-|a| - |b| \leq a + b \leq |a| + |b|$. Thus, $-(|a| + |b|) \leq a + b \leq (|a| + |b|)$. Therefore, $|a + b| \leq |a| + |b|$, by Theorem 2.

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