The Weekly Rigor

No. 161 "A mathematician is a machine for turning coffee into theorems." July 22, 2017

Some Basic Properties of the Additive Inverse
(Part 1)

Reference to the real number axioms and the fundamental properties of addition and multiplication are found in *WR* no. 160. In the following, a, b, and c denote real numbers.

Theorem 1 (Cancellation of Addition): If $a + c = b + c$, then $a = b$.

Preliminary Remark: Note that this theorem is the converse of the Fundamental Addition Property.

Proof: Suppose that

$$a + c = b + c.$$

Hence, there exists $-c$ by Axiom 6. So,

$$(a + c) + (-c) = (b + c) + (-c),$$

by the Fundamental Addition Property. Thus,

$$a + [c + (-c)] = b + [c + (-c)],$$

by Axiom 3. Hence,

$$a + 0 = b + 0,$$

by Axiom 6. Therefore,

$$a = b,$$

by Axiom 5.

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Remark: Without special comment, we shall include in this theorem the variation

If $c + a = c + b$, then $a = b$.

Theorem 2: The additive inverse of every a is unique.

Proof: Suppose that b and c are two (possibly distinct) additive inverses of a. Hence,

$$a + b = 0$$

and

$$a + c = 0,$$

by Axiom 6. So,

$$a + b = a + c.$$

Therefore

$$b = c,$$

by Theorem 1, i.e., the additive inverse of a is unique.

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Theorem 3: If $a + b = 0$, then $b = -a$.

Proof: Suppose that

$$a + b = 0.$$

But

$$a + (-a) = 0,$$

by Axiom 6. Therefore,

$$b = -a,$$

by Theorem 2.

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Theorem 4:

$$a \cdot 0 = 0.$$

Preliminary Remark: In words: The product of any real number times zero is equal to zero.

Proof:

$$a \cdot 0 \overset{\text{A5}}{=} a \cdot (0 + 0) \overset{\text{A4}}{=} a \cdot 0 + a \cdot 0.$$

But

$$-(a \cdot 0)$$

exists, by Axiom 6. Hence,

$$a \cdot 0 + [-(a \cdot 0)] = (a \cdot 0 + a \cdot 0) + [-(a \cdot 0)],$$

by the Fundamental Addition Property. So,

$$a \cdot 0 + [-(a \cdot 0)] = a \cdot 0 + \{a \cdot 0 + [-(a \cdot 0)]\},$$

by Axiom 3. Thus,

$$0 = a \cdot 0 + 0,$$

by Axiom 6. Therefore,

$$0 = a \cdot 0,$$

by Axiom 5.

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