The Weekly Rigor

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Some Basic Properties of the Additive Inverse

(Part 2)

Theorem 5: $$-a = (-1)a.$$

Preliminary Remark: In words: The negative of any real number is equal to negative one times that real number.

Proof: $$0 \overset{\text{T4}}{=} a \cdot 0 \overset{\text{A2}}{=} 0 \cdot a \overset{\text{A6}}{=} [1 + (-1)]a \overset{\text{A4}}{=} 1 \cdot a + (-1)a \overset{\text{A5}}{=} a + (-1)a.$$
Hence,
$$-a + 0 = -a + [a + (-1)a],$$
by the Fundamental Addition Property. So,
$$-a + 0 = (-a + a) + (-1)a,$$
by Axiom 3. Thus,
$$-a = (-a + a) + (-1)a,$$
by Axiom 5. Hence,
$$-a = 0 + (-1)a,$$
by Axiom 6. Therefore,
$$-a = (-1)a,$$
by Axiom 5.

■

Theorem 6: $$-(-a) = a.$$

Preliminary Remark: In words: The double negative of any real number is equal to just the real number alone.

Proof: $$-(-a) + (-a) = 0,$$
by Axiom 6. Hence,
$$[-(-a) + (-a)] + a = 0 + a,$$
by the Fundamental Addition Property. So,
$$-(-a) + (-a + a) = 0 + a,$$
by Axiom 3. Thus,
$$-(-a) + (-a + a) = a,$$
by Axiom 5. Hence,
$$-(-a) + 0 = a,$$
by Axiom 6. Therefore,
$$-(-a) = a,$$
by Axiom 5.

■

Theorem 7: $$a(-b) = -ab.$$

Proof: $$a(-b) \overset{\text{T5}}{=} a[(-1)b] \overset{\text{A3}}{=} [a(-1)]b \overset{\text{A2}}{=} [(-1)a]b \overset{\text{A3}}{=} (-1)(ab) \overset{\text{T5}}{=} -ab.$$

■

Theorem 8: $$(-a)b = -ab.$$

Proof: $$(-a)b \overset{\text{A2}}{=} b(-a) \overset{\text{T7}}{=} -ba \overset{\text{A2}}{=} -ab.$$

■

Theorem 9: $$(-a)(-b) = ab.$$

Proof: $$(-a)(-b) \overset{\text{T8}}{=} -a(-b) \overset{\text{T7}}{=} -[-(ab)] \overset{\text{T6}}{=} ab.$$

■

Theorem 10: $$-(a+b) = (-a) + (-b).$$

Proof: $$-(a+b) \overset{\text{T5}}{=} (-1)(a+b) \overset{\text{A4}}{=} (-1)a + (-1)b \overset{\text{T5}}{=} (-a) + (-b).$$

■

Theorem 11: $a = b$ if and only if $-a = -b$.

Proof: Suppose that

$$a = b.$$

But

$$-a = -a.$$

Hence,

$$-a = -b,$$

by substitution.

Suppose that

$$-a = -b.$$

Hence, by the Fundamental Addition Property,

$$-a + b = -b + b \overset{\text{A6}}{=} 0.$$

So,

$$b = -(-a),$$

by Theorem 3. Therefore,

$$b = a,$$

by Theorem 6.

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