The Weekly Rigor

No. 163 — "A mathematician is a machine for turning coffee into theorems." — August 5, 2017

The Uniqueness of the Additive Identity and Inverse in a Vector Space

Definition 1: A *vector space* over the field $\mathbb{F}$ is a set V along with an addition on V and a scalar multiplication on V such that the following properties hold.

1. commutativity of addition: $\vec{v} + \vec{w} = \vec{w} + \vec{v}$ for all $\vec{v}, \vec{w}$ in V.

2. associativity of addition: $(\vec{v} + \vec{w}) + \vec{y} = \vec{v} + (\vec{w} + \vec{y})$ for all $\vec{v}, \vec{w}, \vec{y}$ in V.

3. associativity of scalar multiplication: $(\lambda\alpha)\vec{v} = \lambda(\alpha\vec{v})$ for all λ, α in $\mathbb{F}$ and $\vec{v}$ in V.

4. the vector space contains an additive identity:
   There exists a $\vec{0}$ in V such that $\vec{0} + \vec{v} = \vec{v}$ for all $\vec{v}$ in V.

5. each vector in the vector space has an additive inverse:
   For each $\vec{v}$ in V there exists a $-\vec{v}$ in V such that $\vec{v} + (-\vec{v}) = \vec{0}$.

6. scaling a vector by the multiplicative identity in the field leaves the vector unchanged.
   $$1\vec{v} = \vec{v} \text{ for all } \vec{v} \text{ in } V.$$

7. distributive properties: for all scalars λ and α in $\mathbb{F}$ and vectors $\vec{v}$ and $\vec{w}$ in V,
   $$\lambda(\vec{v} + \vec{w}) = \lambda\vec{v} + \lambda\vec{w} \quad \text{and} \quad \vec{v}(\lambda + \alpha) = \vec{v}\lambda + \vec{v}\lambda.$$

Theorem 1 (Right Cancellation of Addition): If $\vec{v} + \vec{y} = \vec{w} + \vec{y}$, then $\vec{v} = \vec{w}$.

Proof: Suppose that
$$\vec{v} + \vec{y} = \vec{w} + \vec{y}.$$
Hence, there exists $-\vec{y}$ such that
$$\vec{y} + (-\vec{y}) = \vec{0},$$
by Definition 1.5. So,
$$(\vec{v} + \vec{y}) + (-\vec{y}) = (\vec{w} + \vec{y}) + (-\vec{y}).$$
Thus,
$$\vec{v} + \big(\vec{y} + (-\vec{y})\big) = \vec{w} + \big(\vec{y} + (-\vec{y})\big),$$
by Definition 1.2. Hence,
$$\vec{v} + \vec{0} = \vec{w} + \vec{0},$$

by substitution. Therefore,

$$\vec{v} = \vec{w},$$

by Definition 1.4.

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Theorem 2: The additive identity of a vector space is unique.

Proof: Suppose that $\vec{a}$ and $\vec{b}$ are two (possibly distinct) additive identities for a vector space. Hence,

$$\vec{a} + \vec{v} = \vec{v}$$

and

$$\vec{b} + \vec{v} = \vec{v},$$

by Definition 1.4. So,

$$\vec{a} + \vec{v} = \vec{b} + \vec{v}.$$

Therefore,

$$\vec{a} = \vec{b},$$

by Theorem 1, i.e., the additive inverse of a vector space is unique.

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Theorem 3: The additive inverse of $\vec{v}$ in the vector space V is unique.

Proof: Suppose that $\vec{a}$ and $\vec{b}$ are two (possibly distinct) additive inverses of $\vec{v}$ in the vector space V. Hence,

$$\vec{v} + \vec{a} = \vec{0}$$

and

$$\vec{v} + \vec{b} = \vec{0},$$

by Definition 1.5. So,

$$\vec{v} + \vec{a} = \vec{v} + \vec{b}.$$

Thus,

$$\vec{a} + \vec{v} = \vec{b} + \vec{v},$$

by Definition 1.1. Therefore,

$$\vec{a} = \vec{b}$$

by Theorem 1, i.e., the additive inverse of a vector space is unique.

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