The Weekly Rigor

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A Proof That $i^2 = -1$

INTRODUCTION

Most math textbooks *define* i to be equal to $\sqrt{-1}$. The others begin by defining i^2 to be equal to -1. In fact, the set of complex numbers can be defined in such a way that the fact that $i^2 = -1$ is a proved result and not just a definition. This article lays out one such method.

Definition 1: A *complex number* is an ordered pair (a, b), where $a, b \in \mathbb{R}$. The set of all complex numbers is denoted by $\mathbb{C}$: $\mathbb{C} = \{(a, b): a, b \in \mathbb{R}\}$.

Definition 2: Addition and multiplication on $\mathbb{C}$ is defined by

$$(a, b) + (c, d) = (a + c, b + d)$$
$$(a, b)(c, d) = (ac - bd, ad + bc).$$

Theorem 1: $$(0,1)(0,1) = (-1,0)$$

Proof: $$(0,1)(0,1) \overset{\text{D2}}{=} (0 \cdot 0 - 1 \cdot 1, 0 \cdot 1 + 1 \cdot 0) = (-1,0).$$

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Definition 3: $$(a, b) = a + bi.$$

Remark: We may write the complex numbers $(0, b)$ and $(a, 0)$ as follows:

$$(0, b) = 0 + bi = bi$$
$$(a, 0) = a + 0i = a.$$

Theorem 2: $i^2 = -1.$

Proof: $i^2 = i \cdot i = (1 \cdot i)(1 \cdot i) \overset{\text{D3}}{\cong} (0 + 1i)(0 + 1i) \overset{\text{D3}}{\cong} (0,1)(0,1) \overset{\text{T1}}{\cong} (-1,0) \overset{\text{D3}}{\cong} -1.$ ∎