The Weekly Rigor

No. 183 "A mathematician is a machine for turning coffee into theorems." December 23, 2017

Proofs of the Elementary Properties of Definite Integrals
(Part 1)

Definition 1: $\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x,$

where $\Delta x = \frac{b-a}{n}$.

Theorem 1: $\int_a^a f(x)dx = 0.$

Proof: $\int_a^a f(x)dx \overset{\text{D1}}{\triangleq} \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\left(\frac{a-a}{n}\right) = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)(0) = \lim_{n\to\infty} 0 = 0.$

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Theorem 2: $\int_a^b f(x)dx = -\int_b^a f(x)dx.$

Proof: $\int_a^b f(x)dx \overset{\text{D1}}{\triangleq} \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\left(\frac{b-a}{n}\right) = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\left(\frac{-[a-b]}{n}\right) =$

$= \lim_{n\to\infty} -\sum_{i=1}^{n} f(x_i^*)\left(\frac{a-b}{n}\right) = -\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\left(\frac{a-b}{n}\right) \overset{\text{D1}}{\triangleq} -\int_b^a f(x)dx.$

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Theorem 3: $\int_a^b cf(x)dx = c\int_a^b f(x)dx.$

Proof: $\int_a^b cf(x)dx \overset{\text{D1}}{\triangleq} \lim_{n\to\infty} \sum_{i=1}^{n} cf(x_i^*)\Delta x = \lim_{n\to\infty} c\sum_{i=1}^{n} f(x_i^*)\Delta x = c\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x \overset{\text{D1}}{\triangleq}$

$\overset{\text{D1}}{\triangleq} c\int_a^b f(x)dx.$

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Theorem 4: $\int_a^b c dx = c(b-a)$.

Proof: $\int_a^b c dx \overset{\text{T3}}{\triangleq} c\int_a^b 1 dx \overset{\text{D1}}{\triangleq} c \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{b-a}{n}\right) = c \lim_{n\to\infty} (b-a) \sum_{i=1}^{n} \frac{1}{n} =$
$= c(b-a) \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n} = c(b-a) \lim_{n\to\infty} \frac{n}{n} = c(b-a) \lim_{n\to\infty} 1 = c(b-a)(1) = c(b-a)$.

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Theorem 5: $\int_a^b [f(x) \pm g(x)]dx = \int_b^a f(x)dx \pm \int_b^a g(x)dx$.

Proof: $\int_a^b [f(x) \pm g(x)]dx \overset{\text{D1}}{\triangleq} \lim_{n\to\infty} \sum_{i=1}^{n} [f(x_i^*) \pm g(x_i^*)]\Delta x =$
$= \lim_{n\to\infty} \sum_{i=1}^{n} [f(x_i^*)\Delta x \pm g(x_i^*)\Delta x] = \lim_{n\to\infty} [\sum_{i=1}^{n} f(x_i^*)\Delta x \pm \sum_{i=1}^{n} g(x_i^*)\Delta x] =$
$= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x \pm \lim_{n\to\infty} \sum_{i=1}^{n} g(x_i^*)\Delta x \overset{\text{D1}}{\triangleq} \int_b^a f(x)dx \pm \int_b^a g(x)dx$.

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Theorem 6: If $f(x) \geq 0$ for $a \leq x \leq b$, then $\int_a^b f(x)dx \geq 0$.

Proof: Suppose that $f(x) \geq 0$ for $a \leq x \leq b$. Hence, $\Delta x \geq 0$. So, $\sum_{i=1}^{n} f(x_i^*)\Delta x \geq 0$. Thus, $\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x \geq \lim_{n\to\infty} 0 = 0$. Therefore, $\int_a^b f(x)dx \geq 0$, by Definition 1.

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