The Weekly Rigor

No. 184 "A mathematician is a machine for turning coffee into theorems." December 30, 2017

Proofs of the Elementary Properties of Definite Integrals
(Part 2)

Theorem 7: If $f(x) \geq g(x)$ for $a \leq x \leq b$, then $\int_a^b f(x)dx \geq \int_a^b f(x)dx$.

Proof: Suppose that $f(x) \geq g(x)$ for $a \leq x \leq b$. Hence, $f(x) - g(x) \geq 0$.
So, $\int_a^b [f(x) - g(x)]dx \geq 0$, by Theorem 6. Thus, $\int_b^a f(x)dx - \int_b^a g(x)dx \geq 0$, by Theorem 5.
Therefore, $\int_a^b f(x)dx \geq \int_a^b f(x)dx$

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Theorem 8: If $m \leq f(x) \leq M$ for $a \leq x \leq b$, then $m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$.

Proof: Suppose that $m \leq f(x) \leq M$ for $a \leq x \leq b$. Hence, $\int_a^b m dx \leq \int_a^b f(x)dx \leq \int_a^b M dx$, by Theorem 7. Therefore, $m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$, by Theorem 4.

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Theorem 9: $\left|\int_a^b f(x)dx\right| \leq \int_a^b |f(x)|dx$.

Proof: $-|f(x)| \leq f(x) \leq |f(x)|$, by Theorem 7 of *WR* no. 158. Hence,
$\int_a^b -|f(x)|dx \leq \int_a^b f(x)dx \leq \int_a^b |f(x)|dx$, by Theorem 7. So,
$-\int_a^b |f(x)|dx \leq \int_a^b f(x)dx \leq \int_a^b |f(x)|dx$, by Theorem 3. Therefore,
$\left|\int_a^b f(x)dx\right| \leq \int_a^b |f(x)|dx$, by Theorem 2 of *WR* no. 156.

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Theorem 10: $\int_a^b f(x)dx + \int_b^c f(x)dx = \int_a^c f(x)dx.$

Proof: $\int_a^b f(x)dx + \int_b^c f(x)dx \overset{\text{D1}}{\triangleq} \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\left(\frac{b-a}{n}\right) + \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\left(\frac{c-b}{n}\right) =$
$= \lim_{n\to\infty} \left(\frac{b-a}{n}\right)\sum_{i=1}^{n} f(x_i^*) + \lim_{n\to\infty} \left(\frac{c-b}{n}\right)\sum_{i=1}^{n} f(x_i^*) =$
$= \left(\frac{b-a}{n}\right)\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*) + \left(\frac{c-b}{n}\right)\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*) = \frac{(b-a)+(c-b)}{n}\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*) =$
$= \frac{c-a}{n}\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*) = \lim_{n\to\infty} \left(\frac{c-a}{n}\right)\sum_{i=1}^{n} f(x_i^*) = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\left(\frac{c-a}{n}\right) \overset{\text{D1}}{\triangleq} \int_a^c f(x)dx.$

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