The Weekly Rigor

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The Elementary Properties of Vector Spaces

(Part 1)

INTRODUCTION

In the following, the italicized lower-case Roman letters c and d shall stand for any real numbers (called “scalars”), unless otherwise restricted. The italicized lower-case Roman letters u, v, w, accented by arrows, shall stand for any vectors. Unless otherwise stated, all statements employing such variables shall be taken to hold universally, without exception.

Definition 1: A **vector space** is a nonempty set V of objects, called “vectors,” on which are defined two operations, called “addition” and “multiplication by scalars,” subject to the ten axioms listed below.

1. The sum of $\vec{u}$ and $\vec{v}$, denoted by “$\vec{u} + \vec{v}$,” is in V.

2. $\vec{u} + \vec{v} = \vec{v} + \vec{u}$.

3. $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$.

4. There is a **zero** vector $\vec{0}$ in V such that $\vec{u} + \vec{0} = \vec{u}$.

5. For each $\vec{u}$ in V, there is a vector $-\vec{u}$ in V such that $\vec{u} + (-\vec{u}) = \vec{0}$.

6. The scalar multiple of $\vec{u}$ by c, denoted by “$c\vec{u}$,” is in V.

7. $c(\vec{u} + \vec{v}) = c\vec{u} + c\vec{v}$.

8. $(c + d)\vec{u} = c\vec{u} + d\vec{u}$.

9. $c(d\vec{u}) = (cd)\vec{u}$.

10. $1\vec{u} = \vec{u}$.

Theorem 1: The zero vector is unique.

Proof: Suppose that for $\vec{w}$ in V, $\vec{u}+\vec{w}=\vec{u}$. But $\vec{0}$ is in V, by Axiom 4. Hence, $\vec{0}+\vec{w}=\vec{0}$. Furthermore, $\vec{w}+\vec{0}=\vec{w}$, by Axiom 4. Therefore, $\vec{w} \overset{\text{SUB}}{=} \vec{w}+\vec{0} \overset{\text{A2}}{=} \vec{0}+\vec{w} \overset{\text{SUB}}{=} \vec{0}$.

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Theorem 2: $-\vec{u}$ is the unique vector in V such that $\vec{u}+(-\vec{u})=\vec{0}$.

Proof: Suppose that for $\vec{w}$ in V, $\vec{u}+\vec{w}=\vec{0}$. Hence, $(-\vec{u})+[\vec{u}+\vec{w}]=(-\vec{u})+\vec{0}$. So, $[(-\vec{u})+\vec{u}]+\vec{w}=(-\vec{u})+\vec{0}$, by Axiom 3. Thus, $[\vec{u}+(-\vec{u})]+\vec{w}=(-\vec{u})+\vec{0}$, by Axiom 2. Hence, $\vec{0}+\vec{w}=(-\vec{u})+\vec{0}$, by Axiom 5. So, $\vec{w}+\vec{0}=(-\vec{u})+\vec{0}$, by Axiom 2. Therefore, $\vec{w}=-\vec{u}$, by Axiom 4.

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Theorem 3: $\vec{u}=-(-\vec{u})$.

Proof: $-\vec{u}$ is in V, by Axiom 5. Hence, there is a vector $-(-\vec{u})$ in V such that $-\vec{u}+[-(-\vec{u})]=\vec{0}$, by Axiom 5. So, $\vec{u}+\{-\vec{u}+[-(-\vec{u})]\}=\vec{u}+\vec{0}$. Thus, $[\vec{u}+(-\vec{u})]+[-(-\vec{u})]=\vec{u}+\vec{0}$, by Axiom 3. Hence, $\vec{0}+[-(-\vec{u})]=\vec{u}+\vec{0}$, by Axiom 5. So, $-(-\vec{u})+\vec{0}=\vec{u}+\vec{0}$, by Axiom 2. Therefore, $-(-\vec{u})=\vec{u}$, by Axiom 4.

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Theorem 4: If $\vec{u}+\vec{v}=\vec{u}+\vec{w}$, then $\vec{v}=\vec{w}$.

Proof: Suppose that $\vec{u}+\vec{v}=\vec{u}+\vec{w}$. Hence, $-\vec{u}+(\vec{u}+\vec{v})=-\vec{u}+(\vec{u}+\vec{w})$. So, $(-\vec{u}+\vec{u})+\vec{v}=(-\vec{u}+\vec{u})+\vec{w}$, by Axiom 3. Thus, $[\vec{u}+(-\vec{u})]+\vec{v}=[\vec{u}+(-\vec{u})]+\vec{w}$, by Axiom 2. Hence, $\vec{0}+\vec{v}=\vec{0}+\vec{w}$, by Axiom 5. So, $\vec{v}+\vec{0}=\vec{w}+\vec{0}$, by Axiom 2. Therefore, $\vec{v}=\vec{w}$, by Axiom 4.

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Theorem 5: $0\vec{u}=\vec{0}$.

Proof: $0\vec{u}=(0+0)\vec{u} \overset{\text{A7}}{=} 0\vec{u}+0\vec{u}$. Hence, $0\vec{u}+(-0\vec{u})=[0\vec{u}+0\vec{u}]+(-0\vec{u})$. So, $0\vec{u}+(-0\vec{u})=0\vec{u}+[0\vec{u}+(-0\vec{u})]$, by Axiom 3. Thus, $\vec{0}=0\vec{u}+\vec{0}$, by Axiom 5. Therefore, $\vec{0}=0\vec{u}$, by Axiom 4.

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