The Weekly Rigor

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The Elementary Properties of Vector Spaces
(Part 2)

Theorem 6: $c\vec{0} = \vec{0}$.

Proof: $c\vec{0} \overset{\text{A4}}{=} c(\vec{0} + \vec{0}) \overset{\text{A7}}{=} c\vec{0} + c\vec{0}$. Hence, $c\vec{0} + (-c\vec{0}) = [c\vec{0} + c\vec{0}] + (-c\vec{0})$. So, $c\vec{0} + (-c\vec{0}) = c\vec{0} + [c\vec{0} + (-c\vec{0})]$, by Axiom 3. Thus, $\vec{0} = c\vec{0} + \vec{0}$, by Axiom 5. Therefore, $\vec{0} = c\vec{0}$, by Axiom 4.

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Theorem 7: $(-1)\vec{u} = -\vec{u}$.

Proof: $0\vec{u} = \vec{0}$, by Theorem 5. Hence, $[1 + (-1)]\vec{u} = \vec{0}$. So, $1\vec{u} + (-1)\vec{u} = \vec{0}$, by Axiom 8. Thus, $\vec{u} + (-1)\vec{u} = \vec{0}$, by Axiom 10. Hence, $-\vec{u} + [\vec{u} + (-1)\vec{u}] = -\vec{u} + \vec{0}$. So, $[(-\vec{u}) + \vec{u}] + (-1)\vec{u} = -\vec{u} + \vec{0}$, by Axiom 3. Thus, $[\vec{u} + (-\vec{u})] + (-1)\vec{u} = -\vec{u} + \vec{0}$, by Axiom 2. Hence, $\vec{0} + (-1)\vec{u} = -\vec{u} + \vec{0}$, by Axiom 5. So, $(-1)\vec{u} + \vec{0} = -\vec{u} + \vec{0}$, by Axiom 2. Therefore, $(-1)\vec{u} = -\vec{u}$, by Axiom 4.

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Theorem 8: If $c\vec{u} = \vec{0}$ for some scalar $c \neq 0$, then $\vec{u} = \vec{0}$.

Proof: Suppose that $c\vec{u} = \vec{0}$ for some scalar $c \neq 0$. Hence, $\frac{1}{c}(c\vec{u}) = \frac{1}{c}\vec{0}$. So, $\left(\frac{1}{c}c\right)\vec{u} = \frac{1}{c}\vec{0}$, by Axiom 9. Thus, $1\vec{u} = \frac{1}{c}\vec{0}$. Hence, $\vec{u} = \frac{1}{c}\vec{0}$, by Axiom 10. Therefore, $\vec{u} = \vec{0}$, by Theorem 6.

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Theorem 9: If $c\vec{u} = \vec{0}$, then either $c = 0$ or $\vec{u} = \vec{0}$.

Proof: Suppose that $c\vec{u} = \vec{0}$. Either $c = 0$ or $c \neq 0$.

Case 1: Suppose that $c = 0$. Hence, either $c = 0$ or $\vec{u} = \vec{0}$.

Case 2: Suppose that $c \neq 0$. Hence, $\vec{u} = \vec{0}$, by Theorem 8. So, either $c = 0$ or $\vec{u} = \vec{0}$.

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Theorem 10: If $\vec{u} \neq \vec{0}$, then $c\vec{u} = \vec{0}$ if and only if $c = 0$.

Proof: Suppose that $\vec{u} \neq \vec{0}$.

Suppose that $c\vec{u} = \vec{0}$. Hence, $c = 0$, by Theorem 9.

Suppose that $c = 0$. Hence, $c\vec{u} = \vec{0}$, by Theorem 5.

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Theorem 11: If $\vec{u} \neq \vec{0}$, then $c\vec{u} = d\vec{u}$ if and only if $c = d$.

Proof: Suppose that $\vec{u} \neq \vec{0}$.

Suppose that $c\vec{u} = d\vec{u}$. Hence, $c\vec{u} + (-d\vec{u}) = d\vec{u} + (-d\vec{u})$. So, $c\vec{u} + (-d\vec{u}) = \vec{0}$, by Axiom 5. Thus, $[c + (-d)]\vec{u} = \vec{0}$, by Axiom 8. Hence, $c + (-d) = 0$, by Theorem 10. So, $c = d$.

Suppose that $c = d$. $c\vec{u} = c\vec{u}$. Hence, $c\vec{u} = d\vec{u}$, by substitution.

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