The Weekly Rigor

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A Negative Times a Negative Equals a Positive: An Alternate Proof

This issue offers a proof that a negative number times a negative number equals a positive number, for any real numbers. This proof differs from the one demonstrated as Theorem 9 of *WR* no. 162. Reference to the real number axioms and the Fundamental Multiplication Property are found in *WR* no. 160. In the following, a and b denote real numbers.

Before proceeding to the abstract proof, let us see the reasoning using concrete integers. We claim, say, that $(-4)(-5) = 4 \cdot 5$. We prove this claim as follows.

Claim: $$(-4)(-5) = 4 \cdot 5.$$

Proof: $$5 + (-5) = 0,$$
by Axiom 6. Hence,
$$(-4)\big(5 + (-5)\big) = (-4) \cdot 0,$$
by the Fundamental Multiplication Property. So,
$$(-4)\big(5 + (-5)\big) = 0,$$
by Theorem 4 of *WR* no. 160. Thus,
$$(-4) \cdot 5 + (-4)(-5) = 0,$$
by Axiom 4. Hence,
$$-(4 \cdot 5) + (-4)(-5) = 0,$$
by Theorem 1 of *WR* no. 187. But
$$-(4 \cdot 5) + 4 \cdot 5 = 0,$$
by Axiom 6. Consequently,
$$-(4 \cdot 5) + (-4)(-5) = -(4 \cdot 5) + 4 \cdot 5.$$
Therefore,
$$(-4)(-5) = 4 \cdot 5,$$
by Theorem 1 of *WR* no. 160.

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And now we generalize from the particular numbers 4 and 5 to any real numbers a and b, using exactly the same steps of reasoning.

Theorem 1: $(-a)(-b) = ab.$

Proof: $b + (-b) = 0,$

by Axiom 6. Hence,

$$(-a)\big(b + (-b)\big) = (-a) \cdot 0,$$

by the Fundamental Multiplication Property. So,

$$(-a)\big(b + (-b)\big) = 0,$$

by Theorem 4 of *WR* no. 160. Thus,

$$(-a)b + (-a)(-b) = 0,$$

by Axiom 4. Hence,

$$-(ab) + (-a)(-b) = 0,$$

by Theorem 1 of *WR* no. 187. But

$$-(ab) + ab = 0,$$

by Axiom 6. Consequently,

$$-(ab) + (-a)(-b) = -(ab) + ab.$$

Therefore,

$$(-a)(-b) = ab,$$

by Theorem 1 of *WR* no. 160.

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