The Weekly Rigor

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A Proof of the Zero Product Property

This issue offers a proof of the "Zero Product Property," i.e., the principle that if any two real numbers have a product of 0, then at least one of the numbers is also 0. Reference to the real number axioms and the Fundamental Multiplication Property are found in *WR* no. 160. In the following, a and b denote real numbers.

Before proceeding to the abstract proof, let us see the reasoning using a more concrete situation. We claim, say, that given $4x = 0$, it must be the case that $x = 0$. We prove this claim as follows.

Claim: If $4x = 0$, then $x = 0$.

Proof: Suppose that

$$4x = 0.$$

Hence, there exists $\frac{1}{4}$ such that

$$\left(\frac{1}{4}\right)4 = 1,$$

by Axiom 6. So,

$$\left(\frac{1}{4}\right)4x = \left(\frac{1}{4}\right)\cdot 0,$$

by the Fundamental Multiplication Property. Thus,

$$\left(\frac{1}{4}\right)4x = 0,$$

by Theorem 4 of *WR* no. 161. So,

$$(1)x = 0,$$

by substitution. Therefore,

$$x = 0,$$

by Axiom 5.

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And now we generalize from the particular situation of 4 times an unknown *x* to the product of any real numbers *a* and *b*, using similar steps of reasoning.

Theorem 1: If $ab = 0$, then either $a = 0$ or $b = 0$.

Proof: Suppose that

$$ab = 0.$$

WLOG, suppose that

$$b \neq 0.$$

Hence, there exists $\frac{1}{b}$ such that

$$b\left(\frac{1}{b}\right) = 1,$$

by Axiom 6. So,

$$ab\left(\frac{1}{b}\right) = 0 \cdot \left(\frac{1}{b}\right),$$

by the Fundamental Multiplication Property. Thus,

$$ab\left(\frac{1}{b}\right) = \left(\frac{1}{b}\right) \cdot 0,$$

by Axiom 2. Hence,

$$ab\left(\frac{1}{b}\right) = 0,$$

by Theorem 4 of *WR* no. 161. So,

$$a(1) = 0,$$

by substitution. Thus,

$$a = 0,$$

by Axiom 5. Therefore, either $a = 0$ or $b = 0$.

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“Only he who never plays, never loses.”