The Weekly Rigor

No. 221 “A mathematician is a machine for turning coffee into theorems.” September 15, 2018

20 Problems in Calculating Type 1 Difference Quotients
(Part 2)

SELECTED SOLUTIONS

1. $f(x) = x \quad \Longrightarrow \quad f(x+h) = x+h.$
So, by substitution, $\frac{f(x+h)-f(x)}{h} = \frac{(x+h)-x}{h} = \frac{x+h-x}{h} = \frac{h}{h} = 1.$

3. $f(x) = x^3 \quad \Longrightarrow \quad f(x+h) = (x+h)^3.$
So, by substitution, $\frac{f(x+h)-f(x)}{h} = \frac{(x+h)^3-x^3}{h} = \frac{(x+h)(x+h)^2-x^3}{h} =$
$= \frac{(x+h)[(x+h)(x+h)]-x^3}{h} = \frac{(x+h)(x^2+xh+hx+h^2)-x^2}{h} =$
$= \frac{(x+h)(x^2+2xh+h^2)-x^2}{h} = \frac{x^2(x+h)+2xh(x+h)+h^2(x+h)-x^2}{h} =$
$= \frac{x^3+x^2h+2x^2h+2xh^2+xh^2+h^3-x^3}{h} = \frac{3x^2h+3xh^2+h^3}{h} = \frac{h(3x^2+3xh+h^2)}{h} =$
$= 3x^2+3xh+h^2.$

5. $f(x) = 2x^2+5 \quad \Longrightarrow \quad f(x+h) = 2(x+h)^2+5.$
So, by substitution, $\frac{f(x+h)-f(x)}{h} = \frac{[2(x+h)^2+5]-[2x^2+5]}{h} =$
$= \frac{[2(x+h)(x+h)+5]-[2x^2+5]}{h} = \frac{2(x^2+xh+hx+h^2)+5-2x^2-5}{h} =$
$= \frac{2(x^2+2xh+h^2)+5-2x^2-5}{h} = \frac{2x^2+4xh+2h^2+5-2x^2-5}{h} = \frac{4xh+2h^2}{h} =$
$= \frac{h(4x+2h)}{h} = 4x+2h.$

7. $f(x) = x^3 - 2x^2 + 3 \quad \Longrightarrow \quad f(x+h) = (x+h)^3 - 2(x+h)^2 + 3$

So, by substitution, $\frac{f(x+h)-f(x)}{h} = \frac{[(x+h)^3-2(x+h)^2+3]-[x^3-2x^2+3]}{h} =$

$= \frac{(x+h)(x+h)^2-2(x^2+2xh+h^2)+3-x^3+2x^2-3}{h} =$

$= \frac{(x+h)(x^2+2xh+h^2)-2x^2-4xh-2h^2+3-x^3+2x^2-3}{h} =$

$= \frac{x^2(x+h)+2xh(x+h)+h^2(x+h)-2x^2-4xh-2h^2+3-x^3+2x^2-3}{h} =$

$= \frac{x^3+x^2h+2x^2h+2xh^2+xh^2+h^3-2x^2-4xh-2h^2+3-x^3+2x^2-3}{h} =$

$= \frac{3x^2h+3xh^2+h^3-4xh-2h^2}{h} = \frac{h(3x^2+3xh+h^2-4x-2h)}{h} =$

$= 3x^2 + 3xh + h^2 - 4x - 2h.$

9. $f(x) = mx + b \quad \Longrightarrow \quad f(x+h) = m(x+h) + b.$

So, by substitution, $\frac{f(x+h)-f(x)}{h} = \frac{[m(x+h)+b]-[mx+b]}{h} = \frac{mx+mh+b-mx-b}{h} =$

$= \frac{mh}{h} = m.$

10. $f(x) = ax^2 + bx + c \quad \Longrightarrow \quad f(x+h) = a(x+h)^2 + b(x+h) + c.$

So, by substitution, $\frac{f(x+h)-f(x)}{h} = \frac{[a(x+h)^2+b(x+h)+c]-[ax^2+bx+c]}{h} =$

$= \frac{a(x^2+2xh+h^2)+bx+bh+c-ax^2-bx-c}{h} = \frac{ax^2+2axh+ah^2+bx+bh+c-ax^2-bx-c}{h} =$

$= \frac{2axh+ah^2+bh}{h} = \frac{h(2ax+ah+b)}{h} = 2ax + ah + b.$

11. $f(x) = \frac{1}{x} \quad \Longrightarrow \quad f(x+h) = \frac{1}{x+h}.$

So, by substitution, $\frac{f(x+h)-f(x)}{h} = \frac{\frac{1}{x+h}-\frac{1}{x}}{h} = \frac{\frac{x}{x}\cdot\frac{1}{x+h}-\frac{1}{x}\cdot\frac{x+h}{x+h}}{h} = \frac{\frac{x}{x}\cdot\frac{1}{(x+h)}-\frac{1}{x}\cdot\frac{(x+h)}{(x+h)}}{h} =$

$= \frac{\frac{x-(x+h)}{x(x+h)}}{h} = \frac{\frac{x-x-h}{x(x+h)}}{h} = \frac{\frac{-h}{x(x+h)}}{h} = \frac{\frac{-h}{x(x+h)}}{\frac{h}{1}} = \frac{-h}{x(x+h)}\cdot\frac{1}{h} = \frac{-1}{x(x+h)}$