The Weekly Rigor

No. 222 “A mathematician is a machine for turning coffee into theorems.” September 22, 2018

20 Problems in Calculating Type 1 Difference Quotients
(Part 3)

13. $f(x)=\frac{7}{x+2} \quad \Longrightarrow \quad f(x+h)=\frac{7}{x+h+2}$.

So, by substitution, $\frac{f(x+h)-f(x)}{h}=\frac{\frac{7}{x+h+2}-\frac{7}{x+2}}{h}=$

$$=\frac{\frac{(x+2)}{(x+2)}\cdot\frac{7}{(x+h+2)}-\frac{7}{(x+2)}\cdot\frac{(x+h+2)}{(x+h+2)}}{h}=\frac{\frac{7(x+2)-7(x+h+2)}{(x+2)(x+h+2)}}{h}=\frac{\frac{7x+14-7x-7h-14}{(x+2)(x+h+2)}}{h}=$$

$$=\frac{\frac{-7h}{(x+2)(x+h+2)}}{\frac{1}{h}}=\frac{-7h}{(x+2)(x+h+2)}\cdot\frac{1}{h}=\frac{-7}{(x+2)(x+h+2)}.$$

15. $f(x)=\sqrt{x} \quad \Longrightarrow \quad f(x+h)=\sqrt{x+h}$.

So, by substitution, $\frac{f(x+h)-f(x)}{h}=\frac{\sqrt{x+h}-\sqrt{x}}{h}=$

$$=\frac{\sqrt{x+h}-\sqrt{x}}{h}\cdot\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}=\frac{\left(\sqrt{x+h}-\sqrt{x}\right)}{h}\cdot\frac{\left(\sqrt{x+h}+\sqrt{x}\right)}{\left(\sqrt{x+h}+\sqrt{x}\right)}=$$

$$=\frac{(x+h)+\sqrt{x}\sqrt{x+h}-\sqrt{x}\sqrt{x+h}-x}{h\left(\sqrt{x+h}+\sqrt{x}\right)}=\frac{(x+h)-x}{h\left(\sqrt{x+h}+\sqrt{x}\right)}=\frac{x+h-x}{h\left(\sqrt{x+h}+\sqrt{x}\right)}=$$

$$=\frac{h}{h\left(\sqrt{x+h}+\sqrt{x}\right)}=\frac{1}{\sqrt{x+h}+\sqrt{x}}.$$

17. $f(x)=\sqrt{x^2+1} \quad \Rightarrow \quad f(x+h)=\sqrt{(x+h)^2+1}$.

So, by substitution, $\frac{f(x+h)-f(x)}{h}=\frac{\sqrt{(x+h)^2+1}-\sqrt{x^2+1}}{h}=$

$$=\frac{\left(\sqrt{(x+h)^2+1}-\sqrt{x^2+1}\right)}{h}\cdot\frac{\left(\sqrt{(x+h)^2+1}+\sqrt{x^2+1}\right)}{\left(\sqrt{(x+h)^2+1}+\sqrt{x^2+1}\right)}=\frac{[(x+h)^2+1]-(x^2+1)}{h\left(\sqrt{(x+h)^2+1}+\sqrt{x^2+1}\right)}$$

$$=\frac{x^2+2xh+h^2+1-x^2-1}{h\left(\sqrt{(x+h)^2+1}+\sqrt{x^2+1}\right)}=\frac{2xh+h^2}{h\left(\sqrt{(x+h)^2+1}+\sqrt{x^2+1}\right)}=\frac{h(2x+h)}{h\left(\sqrt{(x+h)^2+1}+\sqrt{x^2+1}\right)}$$

$$=\frac{2x+h}{\sqrt{(x+h)^2+1}+\sqrt{x^2+1}}$$

19. $f(x)=\frac{1}{\sqrt{x}} \quad \Rightarrow \quad f(x+h)=\frac{1}{\sqrt{x+h}}$.

So, by substitution, $\frac{f(x+h)-f(x)}{h}=\frac{\frac{1}{\sqrt{x+h}}-\frac{1}{\sqrt{x}}}{h}=$

$$=\frac{\frac{1}{\sqrt{x+h}}-\frac{1}{\sqrt{x}}}{h}\cdot\frac{\frac{1}{\sqrt{x+h}}+\frac{1}{\sqrt{x}}}{\frac{1}{\sqrt{x+h}}+\frac{1}{\sqrt{x}}}=\frac{\frac{1}{x+h}-\frac{1}{x}}{h\left(\frac{1}{\sqrt{x+h}}+\frac{1}{\sqrt{x}}\right)}=\frac{\frac{x}{x}\cdot\frac{1}{(x+h)}-\frac{1}{x}\cdot\frac{(x+h)}{(x+h)}}{h\left(\frac{1}{\sqrt{x+h}}+\frac{1}{\sqrt{x}}\right)}=\frac{\frac{x-(x+h)}{x(x+h)}}{h\left(\frac{1}{\sqrt{x+h}}+\frac{1}{\sqrt{x}}\right)}=$$

$$=\frac{\frac{\frac{x-x-h}{x(x+h)}}{h}}{h\left(\frac{1}{\sqrt{x+h}}+\frac{1}{\sqrt{x}}\right)}=\frac{\frac{-h}{x(x+h)}}{h\left(\frac{1}{\sqrt{x+h}}+\frac{1}{\sqrt{x}}\right)}=\frac{\frac{-h}{x(x+h)}}{\frac{h}{1}\left(\frac{1}{\sqrt{x+h}}+\frac{1}{\sqrt{x}}\right)}=\frac{\frac{-h}{x(x+h)}\cdot\frac{1}{h}}{\frac{1}{\sqrt{x+h}}+\frac{1}{\sqrt{x}}}=$$

$$=\frac{\frac{-1}{x(x+h)}}{\frac{1}{\sqrt{x+h}}+\frac{1}{\sqrt{x}}}=\frac{\frac{-1}{x(x+h)}}{\frac{\sqrt{x}}{\sqrt{x}}\cdot\frac{1}{\sqrt{x+h}}+\frac{1}{\sqrt{x}}\cdot\frac{\sqrt{x+h}}{\sqrt{x+h}}}=\frac{\frac{-1}{x(x+h)}}{\frac{\sqrt{x}+\sqrt{x+h}}{\sqrt{x}\sqrt{x+h}}}=$$

$$=\frac{-1}{x(x+h)}\cdot\frac{\sqrt{x}\sqrt{x+h}}{\sqrt{x}+\sqrt{x+h}}=\frac{-1}{\sqrt{x}\sqrt{x+h}\left(\sqrt{x}+\sqrt{x+h}\right)}.$$