The Weekly Rigor

No. 224 “A mathematician is a machine for turning coffee into theorems.” October 6, 2018

20 Problems in Calculating Type 2 Difference Quotients
(Part 2)

SELECTED SOLUTIONS

1. $f(x) = x \quad \Longrightarrow \quad f(a) = a \;\;\&\;\; f(a+h) = a+h.$
So, by substitution, $\frac{f(a+h)-f(a)}{h} = \frac{(a+h)-a}{h} = \frac{a+h-a}{h} = \frac{h}{h} = 1.$

3. $f(x) = x^3 \quad \Longrightarrow \quad f(a) = a^3 \;\;\&\;\; f(a+h) = (a+h)^3.$
So, by substitution, $\frac{f(a+h)-f(a)}{h} = \frac{(a+h)^3-a^3}{h} = \frac{(a+h)(a+h)^2-a^3}{h} =$
$= \frac{(a+h)[(a+h)(a+h)]-a^3}{h} = \frac{(a+h)(a^2+ah+ha+h^2)-a^2}{h} =$
$= \frac{(a+h)(a^2+2ah+h^2)-a^2}{h} = \frac{a^2(a+h)+2ah(a+h)+h^2(a+h)-a^2}{h} =$
$= \frac{a^3+a^2h+2a^2h+2ah^2+ah^2+h^3-a^3}{h} = \frac{3a^2h+3ah^2+h^3}{h} = \frac{h(3a^2+3ah+h^2)}{h} =$
$= 3a^2 + 3ah + h^2.$

5. $f(x) = 2x^2 + 5 \quad \Longrightarrow \quad f(a) = 2a^2 + 5 \;\;\&\;\; f(a+h) = 2(a+h)^2 + 5.$
So, by substitution, $\frac{f(a+h)-f(a)}{h} = \frac{[2(a+h)^2+5]-[2a^2+5]}{h} =$
$= \frac{[2(a+h)(a+h)+5]-[2a^2+5]}{h} = \frac{2(a^2+ah+h+h^2)+5-2a^2-5}{h} =$
$= \frac{2(a^2+2ah+h^2)+5-2a^2-5}{h} = \frac{2a^2+4ah+2h^2+5-2a^2-5}{h} = \frac{4ah+2h^2}{h} =$
$= \frac{h(4a+2h)}{h} = 4a + 2h.$

7. $f(x) = x^3 - 2x^2 + 3 \quad \Longrightarrow \quad f(a) = a^3 - 2a^2 + 3$ &
$f(a+h) = (a+h)^3 - 2(a+h)^2 + 3.$

So, by substitution, $\frac{f(a+h)-f(a)}{h} = \frac{[(a+h)^3-2(a+h)^2+3]-[a^3-2a^2+3]}{h} =$

$= \frac{(a+h)(a+h)^2-2(a^2+2ah+h^2)+3-a^3+2a^2-3}{h} =$

$= \frac{(a+h)(a^2+2ah+h^2)-2a^2-4ah-2h^2+3-a^3+2a^2-3}{h} =$

$= \frac{a^2(a+h)+2ah(a+h)+h^2(a+h)-2a^2-4ah-2h^2+3-a^3+2a^2-3}{h} =$

$= \frac{a^3+a^2h+2a^2h+2ah^2+ah^2+h^3-2a^2-4ah-2h^2+3-a^3+2a^2-3}{h} =$

$= \frac{3a^2h+3ah^2+h^3-4ah-2h^2}{h} = \frac{h(3a^2+3ah+h^2-4a-2h)}{h} =$

$= 3a^2 + 3ah + h^2 - 4a - 2h.$

9. $f(x) = mx + b \quad \Longrightarrow \quad f(a) = ma + b$ & $f(a+h) = m(a+h) + b.$

So, by substitution, $\frac{f(a+h)-f(a)}{h} = \frac{[m(a+h)+b]-[ma+b]}{h} = \frac{ma+mh+b-ma-b}{h} =$

$= \frac{mh}{h} = m.$

10. $f(x) = px^2 + qx + r \quad \Longrightarrow \quad f(a) = pa^2 + qa + r$ &
$f(a+h) = p(a+h)^2 + q(a+h) + r.$

So, by substitution, $\frac{f(a+h)-f(a)}{h} = \frac{[p(a+h)^2+q(a+h)+r]-[pa^2+qa+r]}{h} =$

$= \frac{p(a^2+2ah+h^2)+qa+qh+r-pa^2-qa-r}{h} = \frac{pa^2+2pah+ph^2+qa+qh+r-pa^2-qa-r}{h} =$

$= \frac{2pah+ph^2+qh}{h} = \frac{h(2pa+ph+q)}{h} = 2pa + ph + q.$

11. $f(x) = \frac{1}{x} \quad \Longrightarrow \quad f(a) = \frac{1}{a}$ & $f(a+h) = \frac{1}{a+h}.$

So, by substitution, $\frac{f(a+h)-f(a)}{h} = \frac{\frac{1}{a+h}-\frac{1}{a}}{h} = \frac{\frac{a}{a}\cdot\frac{1}{a+h}-\frac{1}{a}\cdot\frac{a+h}{a+h}}{h} = \frac{\frac{a}{a}\cdot\frac{1}{(a+h)}-\frac{1}{a}\cdot\frac{(a+h)}{(a+h)}}{h} =$

$= \frac{\frac{a-(a+h)}{a(a+h)}}{h} = \frac{\frac{a-a-h}{a(a+h)}}{h} = \frac{\frac{-h}{a(a+h)}}{h} = \frac{\frac{-h}{a(a+h)}}{\frac{h}{1}} = \frac{-h}{a(a+h)} \cdot \frac{1}{h} = \frac{-1}{a(a+h)}$