The Weekly Rigor

No. 225 "A mathematician is a machine for turning coffee into theorems." October 13, 2018

20 Problems in Calculating Type 2 Difference Quotients
(Part 3)

13. $f(x) = \frac{7}{x+2} \implies f(a) = \frac{7}{a+2}$ & $f(a+h) = \frac{7}{a+h+2}$.

So, by substitution, $\frac{f(a+h)-f(a)}{h} = \frac{\frac{7}{a+h+2}-\frac{7}{a+2}}{h} =$

$= \frac{\frac{(a+2)}{(a+2)}\cdot\frac{7}{(a+h+2)}-\frac{7}{(a+2)}\cdot\frac{(a+h+2)}{(a+h+2)}}{h} = \frac{\frac{7(a+2)-7(a+h+2)}{(a+2)(a+h+2)}}{h} = \frac{\frac{7a+14-7a-7h-14}{(a+2)(a+h+2)}}{h} =$

$= \frac{\frac{-7h}{(a+2)(a+h+2)}}{\frac{h}{1}} = \frac{-7h}{(a+2)(a+h+2)}\cdot\frac{1}{h} = \frac{-7}{(a+2)(a+h+2)}.$

15. $f(x) = \sqrt{x} \implies f(a) = \sqrt{a}$ & $f(a+h) = \sqrt{a+h}$.

So, by substitution, $\frac{f(a+h)-f(a)}{h} = \frac{\sqrt{a+h}-\sqrt{a}}{h} =$

$= \frac{\sqrt{a+h}-\sqrt{a}}{h}\cdot\frac{\sqrt{a+h}+\sqrt{a}}{\sqrt{a+h}+\sqrt{a}} = \frac{(\sqrt{a+h}-\sqrt{a})}{h}\cdot\frac{(\sqrt{a+h}+\sqrt{a})}{(\sqrt{a+h}+\sqrt{a})} =$

$= \frac{(a+h)+\sqrt{a}\sqrt{a+h}-\sqrt{a}\sqrt{a+h}-a}{h(\sqrt{a+h}+\sqrt{a})} = \frac{(a+h)-a}{h(\sqrt{a+h}+\sqrt{a})} = \frac{a+h-a}{h(\sqrt{a+h}+\sqrt{a})} =$

$= \frac{h}{h(\sqrt{a+h}+\sqrt{a})} = \frac{1}{\sqrt{a+h}+\sqrt{a}}.$

17. $f(x)=\sqrt{x^2+1} \quad \Rightarrow \quad f(a)=\sqrt{a^2+1}$ & $f(a+h)=\sqrt{(a+h)^2+1}$.

So, by substitution, $\frac{f(a+h)-f(a)}{h}=\frac{\sqrt{(a+h)^2+1}-\sqrt{a^2+1}}{h}=$

$$=\frac{\left(\sqrt{(a+h)^2+1}-\sqrt{a^2+1}\right)}{h}\cdot\frac{\left(\sqrt{(a+h)^2+1}+\sqrt{a^2+1}\right)}{\left(\sqrt{(a+h)^2+1}+\sqrt{a^2+1}\right)}=\frac{[(a+h)^2+1]-(a^2+1)}{h\left(\sqrt{(a+h)^2+1}+\sqrt{a^2+1}\right)}=$$

$$=\frac{a^2+2ah+h^2+1-a^2-1}{h\left(\sqrt{(a+h)^2+1}+\sqrt{a^2+1}\right)}=\frac{2ah+h^2}{h\left(\sqrt{(a+h)^2+1}+\sqrt{a^2+1}\right)}=\frac{h(2a+h)}{h\left(\sqrt{(a+h)^2+1}+\sqrt{a^2+1}\right)}=$$

$$=\frac{2a+h}{\sqrt{(a+h)^2+1}+\sqrt{a^2+1}}$$

19. $f(x)=\frac{1}{\sqrt{x}} \quad \Rightarrow \quad f(a)=\frac{1}{\sqrt{a}}$ & $f(a+h)=\frac{1}{\sqrt{a+h}}$.

So, by substitution, $\frac{f(a+h)-f(a)}{h}=\frac{\frac{1}{\sqrt{a+h}}-\frac{1}{\sqrt{a}}}{h}=$

$$=\frac{\frac{1}{\sqrt{a+h}}-\frac{1}{\sqrt{a}}}{h}\cdot\frac{\frac{1}{\sqrt{a+h}}+\frac{1}{\sqrt{a}}}{\frac{1}{\sqrt{a+h}}+\frac{1}{\sqrt{a}}}=\frac{\frac{1}{a+h}-\frac{1}{a}}{h\left(\frac{1}{\sqrt{a+h}}+\frac{1}{\sqrt{a}}\right)}=\frac{\frac{a}{a}\cdot\frac{1}{(a+h)}-\frac{1}{a}\cdot\frac{(a+h)}{(a+h)}}{h\left(\frac{1}{\sqrt{a+h}}+\frac{1}{\sqrt{a}}\right)}=\frac{\frac{a-(a+h)}{a(a+h)}}{h\left(\frac{1}{\sqrt{a+h}}+\frac{1}{\sqrt{a}}\right)}=$$

$$=\frac{\frac{a-a-h}{a(a+h)}}{h\left(\frac{1}{\sqrt{a+h}}+\frac{1}{\sqrt{a}}\right)}=\frac{\frac{-h}{a(a+h)}}{h\left(\frac{1}{\sqrt{a+h}}+\frac{1}{\sqrt{a}}\right)}=\frac{\frac{-h}{a(a+h)}}{\frac{h}{1}\left(\frac{1}{\sqrt{a+h}}+\frac{1}{\sqrt{a}}\right)}=\frac{\frac{-h}{a(a+h)}\cdot\frac{1}{h}}{\frac{1}{\sqrt{a+h}}+\frac{1}{\sqrt{a}}}=$$

$$=\frac{\frac{-1}{a(a+h)}}{\frac{1}{\sqrt{a+h}}+\frac{1}{\sqrt{a}}}=\frac{\frac{-1}{a(a+h)}}{\frac{\sqrt{a}}{\sqrt{a}}\cdot\frac{1}{\sqrt{a+h}}+\frac{1}{\sqrt{a}}\cdot\frac{\sqrt{a+h}}{\sqrt{a+h}}}=\frac{\frac{-1}{a(a+h)}}{\frac{\sqrt{a}+\sqrt{a+h}}{\sqrt{a}\sqrt{a+h}}}=$$

$$=\frac{-1}{a(a+h)}\cdot\frac{\sqrt{a}\sqrt{a+h}}{\sqrt{a}+\sqrt{a+h}}=\frac{-1}{\sqrt{a}\sqrt{a+h}\left(\sqrt{a}+\sqrt{a+h}\right)}.$$