The Weekly Rigor

No. 227 “A mathematician is a machine for turning coffee into theorems.” October 27, 2018

20 Problems in Calculating Type 3 Difference Quotients
(Part 2)

SELECTED SOLUTIONS

1. $f(x) = x \quad \Longrightarrow \quad f(x + \Delta x) = x + \Delta x.$
So, by substitution, $\frac{f(x+\Delta x)-f(x)}{\Delta x} = \frac{(x+\Delta x)-x}{\Delta x} = \frac{x+\Delta x-x}{\Delta x} = \frac{\Delta x}{\Delta x} = 1.$

3. $f(x) = x^3 \quad \Longrightarrow \quad f(x + \Delta x) = (x + \Delta x)^3.$
So, by substitution, $\frac{f(x+\Delta x)-f(x)}{\Delta x} = \frac{(x+\Delta x)^3-x^3}{\Delta x} = \frac{(x+\Delta x)(x+\Delta x)^2-x^3}{\Delta x} =$
$= \frac{(x+\Delta x)[(x+\Delta x)(x+\Delta x)]-x^3}{\Delta x} = \frac{(x+\Delta x)(x^2+x\Delta x+\Delta x x+(\Delta x)^2)-x^2}{\Delta x} =$
$= \frac{(x+\Delta x)(x^2+2x\Delta x+(\Delta x)^2)-x^2}{\Delta x} = \frac{x^2(x+\Delta x)+2x\Delta x(x+\Delta x)+(\Delta x)^2(x+\Delta x)-x^2}{\Delta x} =$
$= \frac{x^3+x^2\Delta x+2x^2\Delta x+2x(\Delta x)^2+x(\Delta x)^2+(\Delta x)^3-x^3}{\Delta x} = \frac{3x^2\Delta x+3x(\Delta x)^2+(\Delta x)^3}{h} =$
$= \frac{\Delta x(3x^2+3x\Delta x+(\Delta x)^2)}{\Delta x} = 3x^2 + 3x\Delta x + (\Delta x)^2.$

5. $f(x) = 2x^2 + 5 \quad \Longrightarrow \quad f(x + \Delta x) = 2(x + \Delta x)^2 + 5.$
So, by substitution, $\frac{f(x+\Delta x)-f(x)}{\Delta x} = \frac{[2(x+\Delta x)^2+5]-[2x^2+5]}{\Delta x} =$
$= \frac{[2(x+\Delta x)(x+\Delta x)+5]-[2x^2+5]}{\Delta x} = \frac{2(x^2+x\Delta x+\Delta x x+(\Delta x)^2)+5-2x^2-5}{\Delta x} =$
$= \frac{2(x^2+2x\Delta x+(\Delta x)^2)+5-2x^2-5}{\Delta x} = \frac{2x^2+4x\Delta x+2(\Delta x)^2+5-2x^2-5}{\Delta x} = \frac{4x\Delta x+2(\Delta x)^2}{\Delta x} =$
$= \frac{\Delta x(4x+2\Delta x)}{\Delta x h} = 4x + 2\Delta x.$

7. $f(x) = x^3 - 2x^2 + 3 \quad \Longrightarrow \quad f(x + \Delta x) = (x + \Delta x)^3 - 2(x + \Delta x)^2 + 3$

So, by substitution, $\frac{f(x+\Delta x)-f(x)}{\Delta x} = \frac{[(x+\Delta x)^3-2(x+\Delta x)^2+3]-[x^3-2x^2+3]}{\Delta x} =$

$= \frac{(x+\Delta x)(x+\Delta x)^2-2(x^2+2x\Delta x+(\Delta x)^2)+3-x^3+2x^2-3}{\Delta x} =$

$= \frac{(x+\Delta x)(x^2+2x\Delta x+(\Delta x)^2)-2x^2-4x\Delta x-2(\Delta x)^2+3-x^3+2x^2-3}{\Delta x} =$

$= \frac{x^2(x+\Delta x)+2x\Delta x(x+\Delta x)+(\Delta x)^2(x+\Delta x)-2x^2-4x\Delta x-2(\Delta x)^2+3-x^3+2x^2-3}{\Delta x} =$

$= \frac{x^3+x^2\Delta x+2x^2\Delta x+2x(\Delta x)^2+x(\Delta x)^2+(\Delta x)^3-2x^2-4x\Delta x-2(\Delta x)^2+3-x^3+2x^2-3}{\Delta x} =$

$= \frac{3x^2\Delta x+3x(\Delta x)^2+(\Delta x)^3-4x\Delta x-2(\Delta x)^2}{\Delta x} = \frac{\Delta x(3x^2+3x\Delta x+(\Delta x)^2-4x-2\Delta x)}{\Delta x} =$

$= 3x^2 + 3x\Delta x + (\Delta x)^2 - 4x - 2\Delta x.$

9. $f(x) = mx + b \quad \Longrightarrow \quad f(x + \Delta x) = m(x + \Delta x) + b.$

So, by substitution, $\frac{f(x+\Delta x)-f(x)}{\Delta x} = \frac{[m(x+\Delta x)+b]-[mx+b]}{\Delta x} = \frac{mx+m\Delta x+b-mx-b}{\Delta x} =$

$= \frac{m\Delta x}{\Delta x} = m.$

10. $f(x) = ax^2 + bx + c \quad \Longrightarrow \quad f(x + \Delta x) = a(x + \Delta x)^2 + b(x + \Delta x) + c.$

So, by substitution, $\frac{f(x+\Delta x)-f(x)}{\Delta x} = \frac{[a(x+\Delta x)^2+b(x+\Delta x)+c]-[ax^2+bx+c]}{\Delta x} =$

$= \frac{a(x^2+2x\Delta x+(\Delta x)^2)+bx+b\Delta x+c-ax^2-bx-c}{\Delta x} =$

$= \frac{ax^2+2ax\Delta x+a(\Delta x)^2+bx+b\Delta x+c-ax^2-bx-c}{\Delta x} = \frac{2ax\Delta x+a(\Delta x)^2+b\Delta x}{\Delta x} =$

$= \frac{\Delta x(2ax+a\Delta x+b)}{\Delta x} = 2ax + a\Delta x + b.$

11. $f(x) = \frac{1}{x} \quad \Longrightarrow \quad f(x + \Delta x) = \frac{1}{x+\Delta x}.$

So, by substitution, $\frac{f(x+\Delta x)-f(x)}{\Delta x} = \frac{\frac{1}{x+\Delta x}-\frac{1}{x}}{\Delta x} = \frac{\frac{x}{x}\cdot\frac{1}{x+\Delta x}-\frac{1}{x}\cdot\frac{x+\Delta x}{x+\Delta x}}{\Delta x} = \frac{\frac{x}{x}\cdot\frac{1}{(x+\Delta x)}-\frac{1}{x}\cdot\frac{(x+\Delta x)}{(x+\Delta x)}}{\Delta x} =$

$= \frac{\frac{x-(x+\Delta x)}{x(x+\Delta x)}}{\Delta x} = \frac{\frac{x-x-\Delta x}{x(x+\Delta x)}}{\Delta x} = \frac{\frac{-\Delta x}{x(x+\Delta x)}}{\Delta x} = \frac{\frac{-\Delta x}{x(x+\Delta x)}}{\frac{\Delta x}{1}} = \frac{-\Delta x}{x(x+\Delta x)} \cdot \frac{1}{\Delta x} = \frac{-1}{x(x+\Delta x)}$