The Weekly Rigor

No. 228 "A mathematician is a machine for turning coffee into theorems." November 3, 2018

20 Problems in Calculating Type 3 Difference Quotients
(Part 3)

13. $f(x) = \frac{7}{x+2} \quad \Longrightarrow \quad f(x+\Delta x) = \frac{7}{x+\Delta x+2}$.

So, by substitution, $\frac{f(x+\Delta x)-f(x)}{\Delta x} = \frac{\frac{7}{x+\Delta x+2}-\frac{7}{x+2}}{\Delta x} =$

$= \frac{\frac{(x+2)}{(x+2)}\cdot\frac{7}{(x+\Delta x+2)}-\frac{7}{(x+2)}\cdot\frac{(x+\Delta x+2)}{(x+\Delta x+2)}}{\Delta x} = \frac{\frac{7(x+2)-7(x+\Delta x+2)}{(x+2)(x+\Delta x+2)}}{\Delta x} = \frac{\frac{7x+14-7x-7\Delta x-14}{(x+2)(x+\Delta x+2)}}{\Delta x} =$

$= \frac{\frac{-7\Delta x}{(x+2)(x+\Delta x+2)}}{\frac{1}{\Delta x}} = \frac{-7\Delta x}{(x+2)(x+\Delta x+2)}\cdot\frac{1}{\Delta x} = \frac{-7}{(x+2)(x+\Delta x+2)}$.

15. $f(x) = \sqrt{x} \quad \Longrightarrow \quad f(x+\Delta x) = \sqrt{x+\Delta x}$.

So, by substitution, $\frac{f(x+\Delta x)-f(x)}{\Delta x} = \frac{\sqrt{x+\Delta x}-\sqrt{x}}{\Delta x} =$

$= \frac{\sqrt{x+\Delta x}-\sqrt{x}}{\Delta x}\cdot\frac{\sqrt{x+\Delta x}+\sqrt{x}}{\sqrt{x+\Delta x}+\sqrt{x}} = \frac{\left(\sqrt{x+\Delta x}-\sqrt{x}\right)}{\Delta x}\cdot\frac{\left(\sqrt{x+\Delta x}+\sqrt{x}\right)}{\left(\sqrt{x+\Delta x}+\sqrt{x}\right)} =$

$= \frac{(x+\Delta x)+\sqrt{x}\sqrt{x+\Delta x}-\sqrt{x}\sqrt{x+\Delta x}-x}{\Delta x\left(\sqrt{x+\Delta x}+\sqrt{x}\right)} = \frac{(x+\Delta x)-x}{\Delta x\left(\sqrt{x+\Delta x}+\sqrt{x}\right)} = \frac{x+\Delta x-x}{\Delta x\left(\sqrt{x+\Delta x}+\sqrt{x}\right)} =$

$= \frac{\Delta x}{\Delta x\left(\sqrt{x+\Delta x}+\sqrt{x}\right)} = \frac{1}{\sqrt{x+\Delta x}+\sqrt{x}}$.

17. $f(x)=\sqrt{x^2+1} \quad \Longrightarrow \quad f(x+\Delta x)=\sqrt{(x+\Delta x)^2+1}.$

So, by substitution, $\frac{f(x+\Delta x)-f(x)}{\Delta x}=\frac{\sqrt{(x+\Delta x)^2+1}-\sqrt{x^2+1}}{\Delta x}=$

$$=\frac{\left(\sqrt{(x+\Delta x)^2+1}-\sqrt{x^2+1}\right)}{\Delta x}\cdot\frac{\left(\sqrt{(x+\Delta x)^2+1}+\sqrt{x^2+1}\right)}{\left(\sqrt{(x+\Delta x)^2+1}+\sqrt{x^2+1}\right)}=\frac{[(x+\Delta x)^2+1]-(x^2+1)}{\Delta x\left(\sqrt{(x+\Delta x)^2+1}+\sqrt{x^2+1}\right)}$$

$$=\frac{x^2+2x\Delta x+(\Delta x)^2+1-x^2-1}{\Delta x\left(\sqrt{(x+\Delta x)^2+1}+\sqrt{x^2+1}\right)}=\frac{2x\Delta x+(\Delta x)^2}{\Delta x\left(\sqrt{(x+\Delta x)^2+1}+\sqrt{x^2+1}\right)}=\frac{\Delta x(2x+\Delta x)}{\Delta x\left(\sqrt{(x+\Delta x)^2+1}+\sqrt{x^2+1}\right)}$$

$$=\frac{2x+\Delta x}{\sqrt{(x+\Delta x)^2+1}+\sqrt{x^2+1}}$$

19. $f(x)=\frac{1}{\sqrt{x}} \quad \Longrightarrow \quad f(x+\Delta x)=\frac{1}{\sqrt{x+\Delta x}}.$

So, by substitution, $\frac{f(x+\Delta x)-f(x)}{\Delta x}=\frac{\frac{1}{\sqrt{x+\Delta x}}-\frac{1}{\sqrt{x}}}{\Delta x}=$

$$=\frac{\frac{1}{\sqrt{x+\Delta x}}-\frac{1}{\sqrt{x}}}{\Delta x}\cdot\frac{\frac{1}{\sqrt{x+\Delta x}}+\frac{1}{\sqrt{x}}}{\frac{1}{\sqrt{x+\Delta x}}+\frac{1}{\sqrt{x}}}=\frac{\frac{1}{x+\Delta x}-\frac{1}{x}}{\Delta x\left(\frac{1}{\sqrt{x+\Delta x}}+\frac{1}{\sqrt{x}}\right)}=\frac{\frac{x}{x}\cdot\frac{1}{(x+\Delta x)}-\frac{1}{x}\cdot\frac{(x+\Delta x)}{(x+\Delta x)}}{\Delta x\left(\frac{1}{\sqrt{x+\Delta x}}+\frac{1}{\sqrt{x}}\right)}=\frac{\frac{x-(x+\Delta x)}{x(x+\Delta x)}}{\Delta x\left(\frac{1}{\sqrt{x+\Delta x}}+\frac{1}{\sqrt{x}}\right)}=$$

$$=\frac{\frac{\frac{x-x-\Delta x}{x(x+\Delta x)}}{\Delta x}}{\Delta x\left(\frac{1}{\sqrt{x+\Delta x}}+\frac{1}{\sqrt{x}}\right)}=\frac{\frac{-\Delta x}{x(x+\Delta x)}}{\Delta x\left(\frac{1}{\sqrt{x+\Delta x}}+\frac{1}{\sqrt{x}}\right)}=\frac{\frac{-\Delta x}{x(x+\Delta x)}}{\frac{\Delta x}{1}\left(\frac{1}{\sqrt{x+\Delta x}}+\frac{1}{\sqrt{x}}\right)}=\frac{\frac{-\Delta x}{x(x+\Delta x)}\cdot\frac{1}{\Delta x}}{\frac{1}{\sqrt{x+\Delta x}}+\frac{1}{\sqrt{x}}}=$$

$$=\frac{\frac{-1}{x(x+\Delta x)}}{\frac{1}{\sqrt{x+\Delta x}}+\frac{1}{\sqrt{x}}}=\frac{\frac{-1}{x(x+\Delta x)}}{\frac{\sqrt{x}}{\sqrt{x}}\cdot\frac{1}{\sqrt{x+\Delta x}}+\frac{1}{\sqrt{x}}\cdot\frac{\sqrt{x+\Delta x}}{\sqrt{x+\Delta x}}}=\frac{\frac{-1}{x(x+\Delta x)}}{\frac{\sqrt{x}+\sqrt{x+\Delta x}}{\sqrt{x}\sqrt{x+\Delta x}}}=$$

$$=\frac{-1}{x(x+\Delta x)}\cdot\frac{\sqrt{x}\sqrt{x+\Delta x}}{\sqrt{x}+\sqrt{x+\Delta x}}=\frac{-1}{\sqrt{x}\sqrt{x+\Delta x}\left(\sqrt{x}+\sqrt{x+\Delta x}\right)}.$$