The Weekly Rigor

No. 230 | "A mathematician is a machine for turning coffee into theorems." | November 17, 2018

20 Problems in Calculating Type 4 Difference Quotients
(Part 2)

SELECTED SOLUTIONS

1. $f(x) = x \quad \Longrightarrow \quad f(a) = a.$
   So, by substitution, $\frac{f(x)-f(a)}{x-a} = \frac{x-a}{x-a} = 1.$

3. $f(x) = x^3 \quad \Longrightarrow \quad f(a) = a^3.$
   So, by substitution, $\frac{f(x)-f(a)}{x-a} = \frac{x^3-a^3}{x-a} = \frac{(x-a)(x^2+ax+a^2)}{x-a} = x^2 + ax + a^2.$

5. $f(x) = 2x^2 + 5 \quad \Longrightarrow \quad f(a) = 2a^2 + 5.$
   So, by substitution, $\frac{f(x)-f(a)}{x-a} = \frac{(2x^2+5)-(2a^2+5)}{x-a} = \frac{2x^2+5-2a^2-5}{x-a} = \frac{2x^2-2a^2}{x-a} =$
   $= \frac{2(x^2-a^2)}{x-a} = \frac{2(x+a)(x-a)}{x-a} = 2(x+a).$

7. $f(x) = x^3 - 2x^2 + 3 \quad \Longrightarrow \quad f(a) = a^3 - 2a^2 + 3.$
   So, by substitution, $\frac{f(x)-f(a)}{x-a} = \frac{(x^3-2x^2+3)-(a^3-2a^2+3)}{x-a} =$
   $= \frac{x^3-2x^2+3-a^3+2a^2-3}{x-a} = \frac{x^3-2x^2-a^3+2a^2}{x-a} = \frac{x^3-a^3-2x^2+2a^2}{x-a} = \frac{x^3-a^3-2(x^2-a^2)}{x-a} =$
   $= \frac{(x-a)(x^2+ax+a^2)-2(x-a)(x+a)}{x-a} = \frac{(x-a)[(x^2+ax+a^2)-2(x+a)]}{x-a} =$
   $= (x^2 + ax + a^2) - 2(x + a) = x^2 + ax + a^2 - 2x - 2a.$

9. $f(x) = mx + b \quad \Longrightarrow \quad f(a) = ma + b.$
   So, by substitution, $\frac{f(x)-f(a)}{x-a} = \frac{[mx+b]-[ma+b]}{x-a} = \frac{mx+b-ma-b}{x-a} = \frac{mx-ma}{x-a} =$
   $= \frac{m(x-a)}{x-a} = m.$

10. $f(x)=px^2+qx+r \quad \Longrightarrow \quad f(a)=pa^2+qa+r.$

So, by substitution, $\frac{f(x)-f(a)}{x-a}=\frac{(px^2+qx+r)-(pa^2+qa+r)}{x-a}=\frac{px^2+qx+r-pa^2-qa-r}{x-a}=$

$=\frac{px^2+qx-pa^2-qa}{x-a}=\frac{px^2-pa^2+qx-qa}{x-a}=\frac{p(x^2-a^2)+q(x-a)}{x-a}=$

$=\frac{p(x-a)(x+a)+q(x-a)}{x-a}=\frac{(x-a)[p(x+a)+q]}{x-a}=p(x+a)+q=\ px+pa+q.$

11. $f(x)=\frac{1}{x} \quad \Longrightarrow \quad f(a)=\frac{1}{a}.$

So, by substitution, $\frac{f(x)-f(a)}{x-a}=\frac{\frac{1}{x}-\frac{1}{a}}{x-a}=\frac{\frac{a}{a}\cdot\frac{1}{x}-\frac{1}{a}\cdot\frac{x}{x}}{x-a}=\frac{\frac{a}{ax}-\frac{x}{ax}}{x-a}=\frac{\frac{a-x}{ax}}{x-a}=$

$=\frac{\frac{a-x}{ax}}{\frac{x-a}{1}}=\frac{a-x}{ax}\cdot\frac{1}{x-a}=\frac{-(x-a)}{ax}\cdot\frac{1}{x-a}=\frac{-1}{ax}$

13. $f(x)=\frac{7}{x+2} \quad \Longrightarrow \quad f(a)=\frac{7}{a+2}.$

So, by substitution, $\frac{f(x)-f(a)}{x-a}=\frac{\frac{7}{x+2}-\frac{7}{a+2}}{x-a}=$

$=\frac{\frac{(a+2)}{(a+2)}\cdot\frac{7}{(x+2)}-\frac{7}{(a+2)}\cdot\frac{(x+2)}{(x+2)}}{x-a}=\frac{\frac{7(a+2)-7(x+2)}{(a+2)(x+2)}}{x-a}=\frac{\frac{7a+14-7x-14}{(a+2)(x+2)}}{x-a}=$

$=\frac{\frac{7a-7x}{(a+2)(x+2)}}{\frac{x-a}{1}}=\frac{7a-7x}{(a+2)(x+2)}\cdot\frac{1}{x-a}=\frac{7(a-x)}{(a+2)(x+2)}\cdot\frac{1}{x-a}=\frac{-7(x-a)}{(a+2)(x+2)}\cdot\frac{1}{x-a}=$

$=\frac{-7}{(a+2)(x+2)}.$