The Weekly Rigor

No. 231 "A mathematician is a machine for turning coffee into theorems." November 24, 2018

20 Problems in Calculating Type 4 Difference Quotients
(Part 3)

15. $f(x)=\sqrt{x} \quad \Longrightarrow \quad f(a)=\sqrt{a}$.

So, by substitution, $\frac{f(x)-f(a)}{x-a}=\frac{\sqrt{x}-\sqrt{a}}{x-a}=$

$=\frac{\sqrt{x}-\sqrt{a}}{x-a}\cdot\frac{\sqrt{x}+\sqrt{a}}{\sqrt{x}+\sqrt{a}}=\frac{(\sqrt{x}-\sqrt{a})}{(x-a)}\cdot\frac{(\sqrt{x}+\sqrt{a})}{(\sqrt{x}+\sqrt{a})}=$

$=\frac{x+\sqrt{x}\sqrt{a}-\sqrt{a}\sqrt{x}-a}{(x-a)(\sqrt{x}+\sqrt{a})}=\frac{x-a}{(x-a)(\sqrt{x}+\sqrt{a})}=\frac{1}{\sqrt{x}+\sqrt{a}}$.

17. $f(x)=\sqrt{x^2+1} \quad \Longrightarrow \quad f(a)=\sqrt{a^2+1}$.

So, by substitution, $\frac{f(x)-f(a)}{x-a}=\frac{\sqrt{x^2+1}-\sqrt{a^2+1}}{x-a}=$

$=\frac{\sqrt{x^2+1}-\sqrt{a^2+1}}{x-a}\cdot\frac{\sqrt{x^2+1}+\sqrt{a^2+1}}{\sqrt{x^2+1}+\sqrt{a^2+1}}=\frac{(\sqrt{x^2+1}-\sqrt{a^2+1})}{(x-a)}\cdot\frac{(\sqrt{x^2+1}+\sqrt{a^2+1})}{(\sqrt{x^2+1}+\sqrt{a^2+1})}=$

$=\frac{(x^2+1)+\sqrt{x^2+1}\sqrt{a^2+1}-\sqrt{a^2+1}\sqrt{x^2+1}-(a^2+1)}{(x-a)(\sqrt{x^2+1}+\sqrt{a^2+1})}=\frac{(x^2+1)-(a^2+1)}{(x-a)(\sqrt{x^2+1}+\sqrt{a^2+1})}=$

$=\frac{x^2+1-a^2-1}{(x-a)(\sqrt{x^2+1}+\sqrt{a^2+1})}=\frac{x^2-a^2}{(x-a)(\sqrt{x^2+1}+\sqrt{a^2+1})}=\frac{(x-a)(x+a)}{(x-a)(\sqrt{x^2+1}+\sqrt{a^2+1})}=$

$=\frac{x+a}{\sqrt{x^2+1}+\sqrt{a^2+1}}$.

19. $f(x)=\frac{1}{\sqrt{x}} \implies f(a)=\frac{1}{\sqrt{a}}$.

So, by substitution, $\frac{f(x)-f(a)}{x-a}=\frac{\frac{1}{\sqrt{x}}-\frac{1}{\sqrt{a}}}{x-a}=$

$$=\frac{\frac{1}{\sqrt{x}}-\frac{1}{\sqrt{a}}}{x-a}\cdot\frac{\frac{1}{\sqrt{x}}+\frac{1}{\sqrt{a}}}{\frac{1}{\sqrt{x}}+\frac{1}{\sqrt{a}}}=\frac{\frac{1}{x}-\frac{1}{a}}{(x-a)\left(\frac{1}{\sqrt{x}}+\frac{1}{\sqrt{a}}\right)}=\frac{\frac{a}{a}\cdot\frac{1}{x}-\frac{1}{a}\cdot\frac{x}{x}}{(x-a)\left(\frac{1}{\sqrt{x}}+\frac{1}{\sqrt{a}}\right)}=\frac{\frac{a-x}{ax}}{(x-a)\left(\frac{1}{\sqrt{x}}+\frac{1}{\sqrt{a}}\right)}=$$

$$=\frac{\frac{a-x}{ax}}{\frac{(x-a)}{1}\left(\frac{1}{\sqrt{x}}+\frac{1}{\sqrt{a}}\right)}=\frac{\frac{a-x}{ax}\cdot\frac{1}{x-a}}{\frac{1}{\sqrt{x}}+\frac{1}{\sqrt{a}}}=\frac{\frac{-(x-a)}{ax}\cdot\frac{1}{x-a}}{\frac{1}{\sqrt{x}}+\frac{1}{\sqrt{a}}}=\frac{\frac{-1}{ax}}{\frac{1}{\sqrt{x}}+\frac{1}{\sqrt{a}}}=\frac{\frac{-1}{ax}}{\frac{\sqrt{a}}{\sqrt{a}}\cdot\frac{1}{\sqrt{x}}+\frac{1}{\sqrt{a}}\cdot\frac{\sqrt{x}}{\sqrt{x}}}=$$

$$=\frac{\frac{-1}{ax}}{\frac{\sqrt{a}+\sqrt{x}}{\sqrt{a}\sqrt{x}}}=\frac{-1}{ax}\cdot\frac{\sqrt{a}\sqrt{x}}{\sqrt{a}+\sqrt{x}}=\frac{-1}{\sqrt{a}\sqrt{x}\left(\sqrt{a}+\sqrt{x}\right)}.$$

20. $f(x)=\frac{1}{\sqrt{x+2}} \implies f(a)=\frac{1}{\sqrt{a+2}}$.

So, by substitution, $\frac{f(x)-f(a)}{x-a}=\frac{\frac{1}{\sqrt{x+2}}-\frac{1}{\sqrt{a+2}}}{x-a}=$

$$=\frac{\frac{1}{\sqrt{x+2}}-\frac{1}{\sqrt{a+2}}}{x-a}\cdot\frac{\frac{1}{\sqrt{x+2}}+\frac{1}{\sqrt{a+2}}}{\frac{1}{\sqrt{x+2}}+\frac{1}{\sqrt{a+2}}}=\frac{\frac{1}{x+2}-\frac{1}{a+2}}{(x-a)\left(\frac{1}{\sqrt{x+2}}+\frac{1}{\sqrt{a+2}}\right)}=\frac{\frac{(a+2)}{(a+2)}\cdot\frac{1}{(x+2)}-\frac{1}{(a+2)}\cdot\frac{(x+2)}{(x+2)}}{(x-a)\left(\frac{1}{\sqrt{x+2}}+\frac{1}{\sqrt{a+2}}\right)}=$$

$$=\frac{\frac{(a+2)-(x+2)}{(a+2)(x+2)}}{(x-a)\left(\frac{1}{\sqrt{x+2}}+\frac{1}{\sqrt{a+2}}\right)}=\frac{\frac{a+2-x-2}{(a+2)(x+2)}}{\frac{(x-a)}{1}\left(\frac{1}{\sqrt{x+2}}+\frac{1}{\sqrt{a+2}}\right)}=\frac{\frac{a-x}{(a+2)(x+2)}\cdot\frac{1}{x-a}}{\frac{1}{\sqrt{x+2}}+\frac{1}{\sqrt{a+2}}}=\frac{\frac{-(x-a)}{(a+2)(x+2)}\cdot\frac{1}{x-a}}{\frac{1}{\sqrt{x+2}}+\frac{1}{\sqrt{a+2}}}=$$

$$=\frac{\frac{-1}{(a+2)(x+2)}}{\frac{1}{\sqrt{x+2}}+\frac{1}{\sqrt{a+2}}}=\frac{\frac{-1}{(a+2)(x+2)}}{\frac{\sqrt{a+2}}{\sqrt{a+2}}\cdot\frac{1}{\sqrt{x+2}}+\frac{1}{\sqrt{a+2}}\cdot\frac{\sqrt{x+2}}{\sqrt{x+2}}}=\frac{\frac{-1}{(a+2)(x+2)}}{\frac{\sqrt{a+2}+\sqrt{x+2}}{\sqrt{a+2}\sqrt{x+2}}}=\frac{-1}{(a+2)(x+2)}\cdot\frac{\sqrt{a+2}\sqrt{x+2}}{\sqrt{a+2}+\sqrt{x+2}}=$$

$$=\frac{-1}{\sqrt{a+2}\sqrt{x+2}\left(\sqrt{a+2}+\sqrt{x+2}\right)}.$$