The Weekly Rigor

No. 233 — "A mathematician is a machine for turning coffee into theorems." — December 8, 2018

20 Problems in Calculating Type 5 Difference Quotients

(Part 2)

SELECTED SOLUTIONS

1. $f(x) = x \quad \Longrightarrow \quad f(a) = a$ & $f(b) = b$.
So, by substitution, $\frac{f(b)-f(a)}{b-a} = \frac{b-a}{b-a} = 1$.

3. $f(x) = x^3 \quad \Longrightarrow \quad f(a) = a^3$ & $f(b) = b^3$.
So, by substitution, $\frac{f(b)-f(a)}{b-a} = \frac{b^3-a^3}{b-a} = \frac{(b-a)(b^2+ab+a^2)}{b-a} = b^2 + ab + a^2$.

5. $f(x) = 2x^2 + 5 \quad \Longrightarrow \quad f(a) = 2a^2 + 5$ & $f(b) = 2b^2 + 5$.
So, by substitution, $\frac{f(b)-f(a)}{b-a} = \frac{(2b^2+5)-(2a^2+5)}{b-a} = \frac{2b^2+5-2a^2-5}{b-a} = \frac{2b^2-2a^2}{b-a} =$
$= \frac{2(b^2-a^2)}{b-a} = \frac{2(b+a)(b-a)}{b-a} = 2(b+a)$.

7. $f(x) = x^3 - 2x^2 + 3 \quad \Longrightarrow \quad f(a) = a^3 - 2a^2 + 3$ & $f(b) = b^3 - 2b^2 + 3$.
So, by substitution, $\frac{f(b)-f(a)}{b-a} = \frac{(b^3-2b^2+3)-(a^3-2a^2+3)}{b-a} =$
$= \frac{b^3-2b^2+3-a^3+2a^2-3}{b-a} = \frac{b^3-2b^2-a^3+2a^2}{b-a} = \frac{b^3-a^3-2b^2+2a^2}{b-a} = \frac{b^3-a^3-2(b^2-a^2)}{b-a} =$
$= \frac{(b-a)(b^2+ab+a^2)-2(b-a)(b+a)}{b-a} = \frac{(b-a)[(b^2+ab+a^2)-2(b+a)]}{b-a} =$
$= (b^2 + ab + a^2) - 2(b + a) = b^2 + ab + a^2 - 2b - 2a$.

9. $f(x) = mx + c \quad \Longrightarrow \quad f(a) = ma + c$ & $f(b) = mb + c$.
So, by substitution, $\frac{f(b)-f(a)}{b-a} = \frac{[mb+c]-[ma+c]}{b-a} = \frac{mb+c-ma-c}{b-a} = \frac{mb-ma}{b-a} =$
$= \frac{m(b-a)}{b-a} = m$.

10. $f(x) = px^2 + qx + r \quad \Longrightarrow \quad f(a) = pa^2 + qa + r \quad \& \quad f(b) = pb^2 + qb + r \quad .$

So, by substitution, $\frac{f(b)-f(a)}{b-a} = \frac{(pb^2+qb+r)-(pa^2+qa+r)}{b-a} = \frac{pb^2+qb+r-pa^2-qa-r}{b-a} =$

$= \frac{pb^2+qb-pa^2-qa}{b-a} = \frac{pb^2-pa^2+qb-qa}{b-a} = \frac{p(b^2-a^2)+q(b-a)}{b-a} =$

$= \frac{p(b-a)(b+a)+q(b-a)}{b-a} = \frac{(b-a)[p(b+a)+q]}{b-a} = p(b+a) + q = \; pb + pa + q.$

11. $f(x) = \frac{1}{x} \quad \Longrightarrow \quad f(a) = \frac{1}{a} \quad \& \quad f(b) = \frac{1}{b}.$

So, by substitution, $\frac{f(b)-f(a)}{b-a} = \frac{\frac{1}{b}-\frac{1}{a}}{b-a} = \frac{\frac{a}{a}\cdot\frac{1}{b}-\frac{1}{a}\cdot\frac{b}{b}}{b-a} = \frac{\frac{a}{ab}-\frac{b}{ab}}{b-a} = \frac{\frac{a-b}{ab}}{b-a} =$

$= \frac{\frac{a-b}{ab}}{\frac{b-a}{1}} = \frac{a-b}{ab} \cdot \frac{1}{b-a} = \frac{-(b-a)}{ab} \cdot \frac{1}{b-a} = \frac{-1}{ab}$

13. $f(x) = \frac{7}{x+2} \quad \Longrightarrow \quad f(a) = \frac{7}{a+2} \quad \& \quad f(b) = \frac{7}{b+2}.$

So, by substitution, $\frac{f(b)-f(a)}{b-a} = \frac{\frac{7}{b+2}-\frac{7}{a+2}}{b-a} =$

$= \frac{\frac{(a+2)}{(a+2)}\cdot\frac{7}{(b+2)}-\frac{7}{(a+2)}\cdot\frac{(b+2)}{(b+2)}}{b-a} = \frac{\frac{7(a+2)-7(b+2)}{(a+2)(b+2)}}{b-a} = \frac{\frac{7a+14-7b-14}{(a+2)(b+2)}}{b-a} =$

$= \frac{\frac{7a-7b}{(a+2)(b+2)}}{\frac{b-a}{1}} = \frac{7a-7b}{(a+2)(b+2)} \cdot \frac{1}{b-a} = \frac{7(a-b)}{(a+2)(b+2)} \cdot \frac{1}{b-a} = \frac{-7(b-a)}{(a+2)(b+2)} \cdot \frac{1}{b-a} =$

$= \frac{-7}{(a+2)(b+2)}.$