The Weekly Rigor

No. 234 “A mathematician is a machine for turning coffee into theorems.” December 15, 2018

20 Problems in Calculating Type 5 Difference Quotients
(Part 3)

15. $f(x) = \sqrt{x} \quad \Longrightarrow \quad f(a) = \sqrt{a}$ & $f(b) = \sqrt{b}$.

So, by substitution, $\frac{f(b)-f(a)}{b-a} = \frac{\sqrt{b}-\sqrt{a}}{b-a} =$

$= \frac{\sqrt{b}-\sqrt{a}}{b-a} \cdot \frac{\sqrt{b}+\sqrt{a}}{\sqrt{b}+\sqrt{a}} = \frac{\left(\sqrt{b}-\sqrt{a}\right)}{(b-a)} \cdot \frac{\left(\sqrt{b}+\sqrt{a}\right)}{\left(\sqrt{b}+\sqrt{a}\right)} =$

$= \frac{b+\sqrt{b}\sqrt{a}-\sqrt{a}\sqrt{b}-a}{(b-a)\left(\sqrt{b}+\sqrt{a}\right)} = \frac{b-a}{(b-a)\left(\sqrt{b}+\sqrt{a}\right)} = \frac{1}{\sqrt{b}+\sqrt{a}}.$

17. $f(x) = \sqrt{x^2+1} \quad \Longrightarrow \quad f(a) = \sqrt{a^2+1}$ & $f(b) = \sqrt{b^2+1}$.

So, by substitution, $\frac{f(b)-f(a)}{b-a} = \frac{\sqrt{b^2+1}-\sqrt{a^2+1}}{b-a} =$

$= \frac{\sqrt{b^2+1}-\sqrt{a^2+1}}{b-a} \cdot \frac{\sqrt{b^2+1}+\sqrt{a^2+1}}{\sqrt{b^2+1}+\sqrt{a^2+1}} = \frac{\left(\sqrt{b^2+1}-\sqrt{a^2+1}\right)}{(b-a)} \cdot \frac{\left(\sqrt{b^2+1}+\sqrt{a^2+1}\right)}{\left(\sqrt{b^2+1}+\sqrt{a^2+1}\right)} =$

$= \frac{(b^2+1)+\sqrt{b^2+1}\sqrt{a^2+1}-\sqrt{a^2+1}\sqrt{b^2+1}-(a^2+1)}{(b-a)\left(\sqrt{b^2+1}+\sqrt{a^2+1}\right)} = \frac{(b^2+1)-(a^2+1)}{(b-a)\left(\sqrt{b^2+1}+\sqrt{a^2+1}\right)} =$

$= \frac{b^2+1-a^2-1}{(b-a)\left(\sqrt{b^2+1}+\sqrt{a^2+1}\right)} = \frac{b^2-a^2}{(b-a)\left(\sqrt{b^2+1}+\sqrt{a^2+1}\right)} = \frac{(b-a)(b+a)}{(b-a)\left(\sqrt{b^2+1}+\sqrt{a^2+1}\right)} =$

$= \frac{b+a}{\sqrt{b^2+1}+\sqrt{a^2+1}}.$

19. $f(x)=\frac{1}{\sqrt{x}} \quad \Rightarrow \quad f(a)=\frac{1}{\sqrt{a}} \quad \& \quad f(b)=\frac{1}{\sqrt{b}}$.

So, by substitution, $\frac{f(b)-f(a)}{b-a}=\frac{\frac{1}{\sqrt{b}}-\frac{1}{\sqrt{a}}}{b-a}=$

$$=\frac{\frac{1}{\sqrt{b}}-\frac{1}{\sqrt{a}}}{b-a}\cdot\frac{\frac{1}{\sqrt{b}}+\frac{1}{\sqrt{a}}}{\frac{1}{\sqrt{b}}+\frac{1}{\sqrt{a}}}=\frac{\frac{1}{b}-\frac{1}{a}}{(b-a)\left(\frac{1}{\sqrt{b}}+\frac{1}{\sqrt{a}}\right)}=\frac{\frac{a}{a}\cdot\frac{1}{b}-\frac{1}{a}\cdot\frac{b}{b}}{(b-a)\left(\frac{1}{\sqrt{b}}+\frac{1}{\sqrt{a}}\right)}=\frac{\frac{a-b}{ab}}{(b-a)\left(\frac{1}{\sqrt{b}}+\frac{1}{\sqrt{a}}\right)}=$$

$$=\frac{\frac{a-b}{ab}}{\frac{(b-a)}{1}\left(\frac{1}{\sqrt{b}}+\frac{1}{\sqrt{a}}\right)}=\frac{\frac{a-b}{ab}\cdot\frac{1}{b-a}}{\frac{1}{\sqrt{b}}+\frac{1}{\sqrt{a}}}=\frac{\frac{-(b-a)}{ab}\cdot\frac{1}{b-a}}{\frac{1}{\sqrt{b}}+\frac{1}{\sqrt{a}}}=\frac{\frac{-1}{ab}}{\frac{1}{\sqrt{b}}+\frac{1}{\sqrt{a}}}=\frac{\frac{-1}{ab}}{\frac{\sqrt{a}}{\sqrt{a}}\cdot\frac{1}{\sqrt{b}}+\frac{1}{\sqrt{a}}\cdot\frac{\sqrt{b}}{\sqrt{b}}}=$$

$$=\frac{\frac{-1}{ab}}{\frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}\sqrt{b}}}=\frac{-1}{ab}\cdot\frac{\sqrt{a}\sqrt{b}}{\sqrt{a}+\sqrt{b}}=\frac{-1}{\sqrt{a}\sqrt{b}\left(\sqrt{a}+\sqrt{b}\right)}.$$

20. $f(x)=\frac{1}{\sqrt{x+2}} \quad \Rightarrow \quad f(a)=\frac{1}{\sqrt{a+2}} \quad \& \quad f(b)=\frac{1}{\sqrt{b+2}}$.

So, by substitution, $\frac{f(b)-f(a)}{b-a}=\frac{\frac{1}{\sqrt{b+2}}-\frac{1}{\sqrt{a+2}}}{b-a}=$

$$=\frac{\frac{1}{\sqrt{b+2}}-\frac{1}{\sqrt{a+2}}}{b-a}\cdot\frac{\frac{1}{\sqrt{b+2}}+\frac{1}{\sqrt{a+2}}}{\frac{1}{\sqrt{b+2}}+\frac{1}{\sqrt{a+2}}}=\frac{\frac{1}{b+2}-\frac{1}{a+2}}{(b-a)\left(\frac{1}{\sqrt{b+2}}+\frac{1}{\sqrt{a+2}}\right)}=\frac{\frac{(a+2)}{(a+2)}\cdot\frac{1}{(b+2)}-\frac{1}{(a+2)}\cdot\frac{(b+2)}{(b+2)}}{(b-a)\left(\frac{1}{\sqrt{b+2}}+\frac{1}{\sqrt{a+2}}\right)}=$$

$$=\frac{\frac{(a+2)-(b+2)}{(a+2)(b+2)}}{(b-a)\left(\frac{1}{\sqrt{b+2}}+\frac{1}{\sqrt{a+2}}\right)}=\frac{\frac{a+2-b-2}{(a+2)(b+2)}}{\frac{(b-a)}{1}\left(\frac{1}{\sqrt{b+2}}+\frac{1}{\sqrt{a+2}}\right)}=\frac{\frac{a-b}{(a+2)(b+2)}\cdot\frac{1}{b-a}}{\frac{1}{\sqrt{b+2}}+\frac{1}{\sqrt{a+2}}}=\frac{\frac{-(b-a)}{(a+2)(b+2)}\cdot\frac{1}{b-a}}{\frac{1}{\sqrt{b+2}}+\frac{1}{\sqrt{a+2}}}=$$

$$=\frac{\frac{-1}{(a+2)(b+2)}}{\frac{1}{\sqrt{b+2}}+\frac{1}{\sqrt{a+2}}}=\frac{\frac{-1}{(a+2)(b+2)}}{\frac{\sqrt{a+2}}{\sqrt{a+2}}\cdot\frac{1}{\sqrt{b+2}}+\frac{1}{\sqrt{a+2}}\cdot\frac{\sqrt{b+2}}{\sqrt{b+2}}}=\frac{\frac{-1}{(a+2)(b+2)}}{\frac{\sqrt{a+2}+\sqrt{b+2}}{\sqrt{a+2}\sqrt{b+2}}}=\frac{-1}{(a+2)(b+2)}\cdot\frac{\sqrt{a+2}\sqrt{b+2}}{\sqrt{a+2}+\sqrt{b+2}}=$$

$$=\frac{-1}{\sqrt{a+2}\sqrt{b+2}\left(\sqrt{a+2}+\sqrt{b+2}\right)}.$$