The Weekly Rigor

No. 275 "A mathematician is a machine for turning coffee into theorems." September 28, 2019

12 Problems Solving Composite Trigonometric Equations (Type I)

(Part 2)

SELECTED SOLUTIONS

1. $4\sin\left(\frac{1}{2}x\right) - 2 = 0 \quad \Longrightarrow \quad \sin\left(\frac{1}{2}x\right) = \frac{2}{4} = \frac{1}{2}$. According to *WR* no. 265, problem 1, $\sin(\theta) = \frac{1}{2}$ for $\theta = \frac{\pi}{6}$ and $\theta = \frac{5\pi}{6}$.

Regarding $\theta = \frac{\pi}{6}$, set $\frac{1}{2}x = \theta$. Hence, $\frac{1}{2}x = \frac{\pi}{6}$. Solving for *x*, we have $x = \frac{2\pi}{6} = \frac{\pi}{3}$. Perhaps $\theta + 2\pi = \frac{\pi}{6} + 2\pi$ will also provide a basis for finding solutions for *x*. $\frac{\pi}{6} + 2\pi = \frac{\pi}{6} + \frac{12\pi}{6} = \frac{13\pi}{6}$. Setting $\frac{1}{2}x = \frac{13\pi}{6}$ and solving for *x*, we have $x = \frac{26\pi}{6} = \frac{13\pi}{3}$. But $\frac{13\pi}{3} > \frac{6\pi}{3} = 2\pi$ is outside the interval $[0,2\pi)$.

Regarding $\theta = \frac{5\pi}{6}$, set $\frac{1}{2}x = \theta$. Hence, $\frac{1}{2}x = \frac{5\pi}{6}$. Solving for *x*, we have $x = \frac{10\pi}{6} = \frac{5\pi}{3}$. Perhaps $\theta + 2\pi = \frac{5\pi}{6} + 2\pi$ will also provide a basis for finding solutions for *x*. $\frac{5\pi}{6} + 2\pi = \frac{5\pi}{6} + \frac{12\pi}{6} = \frac{17\pi}{6}$. Setting $\frac{1}{2}x = \frac{17\pi}{6}$ and solving for *x*, we have $x = \frac{34\pi}{6} = \frac{17\pi}{3}$. But $\frac{17\pi}{3} > \frac{6\pi}{3} = 2\pi$ is outside the interval $[0,2\pi)$.

Therefore, the only solutions for *x* are $\frac{\pi}{3}, \frac{5\pi}{3}$.

Check: $4\sin\left(\frac{1}{2}\cdot\frac{\pi}{3}\right) - 2 = 4\sin\left(\frac{\pi}{6}\right) - 2 = 4\cdot\frac{1}{2} - 2 = 2 - 2 = 0$. ✓
$4\sin\left(\frac{1}{2}\cdot\frac{5\pi}{3}\right) - 2 = 4\sin\left(\frac{5\pi}{6}\right) - 2 = 4\cdot\frac{1}{2} - 2 = 2 - 2 = 0$. ✓

3. $\sqrt{2}\sin(2x) + 1 = 0 \quad \Longrightarrow \quad \sin(2x) = \frac{-1}{\sqrt{2}}$. According to *WR* no. 265, problem 7, $\sin(\theta) = \frac{-1}{\sqrt{2}}$ for $\theta = \frac{5\pi}{4}$ and $\theta = \frac{7\pi}{4}$.

Regarding $\theta = \frac{5\pi}{4}$, set $2x = \theta$. Hence, $2x = \frac{5\pi}{4}$. Solving for *x*, we have $x = \frac{5\pi}{8}$. Perhaps $\theta + 2\pi = \frac{5\pi}{4} + 2\pi$ will also provide a basis for finding solutions for *x*. $\frac{5\pi}{4} + 2\pi = \frac{5\pi}{4} + \frac{8\pi}{4} = \frac{13\pi}{4}$. Setting $2x = \frac{13\pi}{4}$ and solving for *x*, we have $x = \frac{13\pi}{8}$. Perhaps $\theta + 4\pi = \frac{5\pi}{4} + 4\pi$ will also provide a basis for finding solutions for *x*. $\frac{5\pi}{4} + 4\pi = \frac{5\pi}{4} + \frac{16\pi}{4} = \frac{21\pi}{4}$. Setting $2x = \frac{21\pi}{4}$ and solving for *x*, we have $x = \frac{21\pi}{8}$. But $\frac{21\pi}{8} > \frac{16\pi}{8} = 2\pi$ is outside the interval $[0,2\pi)$.

Regarding $\theta = \frac{7\pi}{4}$, set $2x = \theta$. Hence, $2x = \frac{7\pi}{4}$. Solving for *x*, we have $x = \frac{7\pi}{8}$. Perhaps $\theta + 2\pi = \frac{7\pi}{4} + 2\pi$ will also provide a basis for finding solutions for *x*.
$\frac{7\pi}{4} + 2\pi = \frac{7\pi}{4} + \frac{8\pi}{4} = \frac{15\pi}{4}$. Setting $2x = \frac{15\pi}{4}$ and solving for *x*, we have $x = \frac{15\pi}{8}$.
Perhaps $\theta + 4\pi = \frac{7\pi}{4} + 4\pi$ will also provide a basis for finding solutions for *x*.
$\frac{7\pi}{4} + 4\pi = \frac{7\pi}{4} + \frac{16\pi}{4} = \frac{23\pi}{4}$. Setting $2x = \frac{23\pi}{4}$ and solving for *x*, we have $x = \frac{23\pi}{8}$.
But $\frac{23\pi}{8} > \frac{16\pi}{8} = 2\pi$ is outside the interval $[0,2\pi)$.

Therefore, the only solutions for *x* are $\frac{5\pi}{8}$, $\frac{7\pi}{8}$, $\frac{13\pi}{8}$, $\frac{15\pi}{8}$.

Check: $\sqrt{2}\sin\left(2 \cdot \frac{5\pi}{8}\right) + 1 = \sqrt{2}\sin\left(\frac{5\pi}{4}\right) + 1 = \sqrt{2} \cdot \frac{-1}{\sqrt{2}} + 1 = -1 + 1 = 0.$✓
$\sqrt{2}\sin\left(2 \cdot \frac{7\pi}{8}\right) + 1 = \sqrt{2}\sin\left(\frac{7\pi}{4}\right) + 1 = \sqrt{2} \cdot \frac{-1}{\sqrt{2}} + 1 = -1 + 1 = 0.$✓
$\sqrt{2}\sin\left(2 \cdot \frac{13\pi}{8}\right) + 1 = \sqrt{2}\sin\left(\frac{13\pi}{4}\right) + 1 = \sqrt{2}\sin\left(\frac{5\pi}{4}\right) + 1 = \sqrt{2} \cdot \frac{-1}{\sqrt{2}} + 1 = -1 + 1 = 0.$✓
$\sqrt{2}\sin\left(2 \cdot \frac{15\pi}{8}\right) + 1 = \sqrt{2}\sin\left(\frac{15\pi}{4}\right) + 1 = \sqrt{2}\sin\left(\frac{7\pi}{4}\right) + 1 = \sqrt{2} \cdot \frac{-1}{\sqrt{2}} + 1 = -1 + 1 = 0.$✓

12. $\sin\left(\frac{x}{2}\right) - 1 = 0 \quad \Longrightarrow \quad \sin\left(\frac{x}{2}\right) = 1$. According to *WR* no. 266, problem 13, $\sin(\theta) = 1$ for $\theta = \frac{\pi}{2}$.

Regarding $\theta = \frac{\pi}{2}$, set $\frac{x}{2} = \theta$. Hence, $\frac{x}{2} = \frac{\pi}{2}$. Solving for *x*, we have $x = \frac{2\pi}{2} = \pi$. Perhaps $\theta + 2\pi = \frac{\pi}{2} + 2\pi$ will also provide a basis for finding solutions for *x*.
$\frac{\pi}{2} + 2\pi = \frac{\pi}{2} + \frac{4\pi}{2} = \frac{5\pi}{2}$. Setting $\frac{x}{2} = \frac{5\pi}{2}$ and solving for *x*, we have $x = \frac{10\pi}{2} = 5\pi$.
But $5\pi > 2\pi$ is outside the interval $[0,2\pi)$.

Therefore, the only solution for *x* is π.

Check: $\sin\left(\frac{\pi}{2}\right) - 1 = 1 - 1 = 0.$✓