The Weekly Rigor

No. 277 "A mathematician is a machine for turning coffee into theorems." October 12, 2019

12 Problems Solving Composite Trigonometric Equations (Type II)

(Part 2)

SELECTED SOLUTIONS

1. $\csc\left(\frac{x}{3}\right) - 2 = 0 \quad \Longrightarrow \quad \csc\left(\frac{x}{3}\right) = 2 \quad \Longrightarrow \quad \frac{1}{\sin\left(\frac{x}{3}\right)} = 2 \quad \Longrightarrow \quad \sin\left(\frac{x}{3}\right) = \frac{1}{2}.$

According to *WR* no. 265, problem 1, $\sin(\theta) = \frac{1}{2}$ for $\theta = \frac{\pi}{6}$ and $\theta = \frac{5\pi}{6}$.

Regarding $\theta = \frac{\pi}{6}$, set $\frac{x}{3} = \theta$. Hence, $\frac{x}{3} = \frac{\pi}{6}$. Solving for *x*, we have $x = \frac{3\pi}{6} = \frac{\pi}{2}$. Perhaps $\theta + 2\pi = \frac{\pi}{6} + 2\pi$ will also provide a basis for finding solutions for *x*.
$\frac{\pi}{6} + 2\pi = \frac{\pi}{6} + \frac{12\pi}{6} = \frac{13\pi}{6}$. Setting $\frac{x}{3} = \frac{13\pi}{6}$ and solving for *x*, we have $x = \frac{39\pi}{6} = \frac{13\pi}{2}$.
But $\frac{13\pi}{2} > \frac{4\pi}{2} = 2\pi$ is outside the interval $[0,2\pi)$.

Therefore, the only solution for *x* is $\frac{\pi}{2}$.

Check: $\csc\left(\frac{1}{3}\cdot\frac{\pi}{2}\right) - 2 = \csc\left(\frac{\pi}{6}\right) - 2 = \frac{1}{\sin\left(\frac{\pi}{6}\right)} - 2 = \frac{1}{\left(\frac{1}{2}\right)} - 2 = 2 - 2 = 0.$ ✓

3. $\sqrt{3}\sec(2x) - 2 = 0 \quad \Longrightarrow \quad \sec(2x) = \frac{2}{\sqrt{3}} \quad \Longrightarrow \quad \frac{1}{\cos(2x)} = \frac{2}{\sqrt{3}} \quad \Longrightarrow \quad \cos(2x) = \frac{\sqrt{3}}{2}.$

According to *WR* no. 268, problem 28 $\cos(\theta) = \frac{\sqrt{3}}{2}$ for $\theta = \frac{\pi}{6}$ and $\theta = \frac{11\pi}{6}$.

Regarding $\theta = \frac{\pi}{6}$, set $2x = \theta$. Hence, $2x = \frac{\pi}{6}$. Solving for *x*, we have $x = \frac{\pi}{12}$.
Perhaps $\theta + 2\pi = \frac{\pi}{6} + 2\pi$ will also provide a basis for finding solutions for *x*.
$\frac{\pi}{6} + 2\pi = \frac{\pi}{6} + \frac{12\pi}{6} = \frac{13\pi}{6}$. Setting $2x = \frac{13\pi}{6}$ and solving for *x*, we have $x = \frac{13\pi}{12}$.
Perhaps $\theta + 4\pi = \frac{\pi}{6} + 4\pi$ will also provide a basis for finding solutions for *x*.
$\frac{\pi}{6} + 4\pi = \frac{\pi}{6} + \frac{24\pi}{6} = \frac{25\pi}{6}$. Setting $2x = \frac{25\pi}{6}$ and solving for *x*, we have $x = \frac{25\pi}{12}$.
But $\frac{25\pi}{12} > \frac{24\pi}{12} = 2\pi$ is outside the interval $[0,2\pi)$.

Regarding $\theta = \frac{11\pi}{6}$, set $2x = \theta$. Hence, $2x = \frac{11\pi}{6}$. Solving for x, we have $x = \frac{11\pi}{12}$. Perhaps $\theta + 2\pi = \frac{11\pi}{6} + 2\pi$ will also provide a basis for finding solutions for x. $\frac{11\pi}{6} + 2\pi = \frac{11\pi}{6} + \frac{12\pi}{6} = \frac{23\pi}{6}$. Setting $2x = \frac{23\pi}{6}$ and solving for x, we have $x = \frac{23\pi}{12}$. Perhaps $\theta + 4\pi = \frac{11\pi}{6} + 4\pi$ will also provide a basis for finding solutions for x. $\frac{11\pi}{6} + 4\pi = \frac{11\pi}{6} + \frac{24\pi}{6} = \frac{35\pi}{6}$. Setting $2x = \frac{35\pi}{6}$ and solving for x, we have $x = \frac{35\pi}{12}$. But $\frac{35\pi}{12} > \frac{24\pi}{12} = 2\pi$ is outside the interval $[0,2\pi)$.

Therefore, the only solutions for x are $\frac{\pi}{12}, \frac{11\pi}{12}, \frac{13\pi}{12}, \frac{23\pi}{12}$.

Check: $\sqrt{3}\sec\left(2 \cdot \frac{\pi}{12}\right) - 2 = \sqrt{3}\sec\left(\frac{\pi}{6}\right) - 2 = \frac{\sqrt{3}}{\cos\left(\frac{\pi}{6}\right)} - 2 = \frac{\sqrt{3}}{\left(\frac{\sqrt{3}}{2}\right)} - 2 = 2 - 2 = 0.$ ✓

$\sqrt{3}\sec\left(2 \cdot \frac{11\pi}{12}\right) - 2 = \sqrt{3}\sec\left(\frac{11\pi}{6}\right) - 2 = \frac{\sqrt{3}}{\cos\left(\frac{11\pi}{6}\right)} - 2 = \frac{\sqrt{3}}{\left(\frac{\sqrt{3}}{2}\right)} - 2 = 0.$ ✓

$\sqrt{3}\sec\left(2 \cdot \frac{13\pi}{12}\right) - 2 = \sqrt{3}\sec\left(\frac{13\pi}{6}\right) - 2 = \frac{\sqrt{3}}{\cos\left(\frac{13\pi}{6}\right)} - 2 = \frac{\sqrt{3}}{\cos\left(\frac{\pi}{6}\right)} - 2 = \frac{\sqrt{3}}{\left(\frac{\sqrt{3}}{2}\right)} - 2 = 0.$ ✓

$\sqrt{3}\sec\left(2 \cdot \frac{23\pi}{12}\right) - 2 = \sqrt{3}\sec\left(\frac{23\pi}{6}\right) - 2 = \frac{\sqrt{3}}{\cos\left(\frac{23\pi}{6}\right)} - 2 = \frac{\sqrt{3}}{\cos\left(\frac{11\pi}{6}\right)} - 2 = \frac{\sqrt{3}}{\left(\frac{\sqrt{3}}{2}\right)} - 2 = 0.$ ✓