The Weekly Rigor

No. 278 "A mathematician is a machine for turning coffee into theorems." October 19, 2019

12 Problems Solving Composite Trigonometric Equations (Type II)

(Part 3)

6. $\csc(3x) + \sqrt{2} = 0 \quad \Rightarrow \quad \csc(3x) = -\sqrt{2} \quad \Rightarrow \quad \frac{1}{\sin(3x)} = -\sqrt{2} \quad \Rightarrow \quad \sin(3x) = \frac{-1}{\sqrt{2}}$.
According to *WR* no. 265, problem 7, $\sin(\theta) = \frac{-1}{\sqrt{2}}$ for $\theta = \frac{5\pi}{4}$ and $\theta = \frac{7\pi}{4}$.

Regarding $\theta = \frac{5\pi}{4}$, set $3x = \theta$. Hence, $3x = \frac{5\pi}{4}$. Solving for *x*, we have $x = \frac{5\pi}{12}$.
Perhaps $\theta + 2\pi = \frac{5\pi}{4} + 2\pi$ will also provide a basis for finding solutions for *x*.
$\frac{5\pi}{4} + 2\pi = \frac{5\pi}{4} + \frac{8\pi}{4} = \frac{13\pi}{4}$. Setting $3x = \frac{13\pi}{4}$ and solving for *x*, we have $x = \frac{13\pi}{12}$.
Perhaps $\theta + 4\pi = \frac{5\pi}{4} + 4\pi$ will also provide a basis for finding solutions for *x*.
$\frac{5\pi}{4} + 4\pi = \frac{5\pi}{4} + \frac{16\pi}{4} = \frac{21\pi}{4}$. Setting $3x = \frac{21\pi}{4}$ and solving for *x*, we have $x = \frac{21\pi}{12}$.
Perhaps $\theta + 6\pi = \frac{5\pi}{4} + 6\pi$ will also provide a basis for finding solutions for *x*.
$\frac{5\pi}{4} + 6\pi = \frac{5\pi}{4} + \frac{24\pi}{4} = \frac{29\pi}{4}$. Setting $3x = \frac{29\pi}{4}$ and solving for *x*, we have $x = \frac{29\pi}{12}$.
But $\frac{29\pi}{12} > \frac{24\pi}{12} = 2\pi$ is outside the interval $[0,2\pi)$.

Regarding $\theta = \frac{7\pi}{4}$, set $3x = \theta$. Hence, $3x = \frac{7\pi}{4}$. Solving for *x*, we have $x = \frac{7\pi}{12}$.
Perhaps $\theta + 2\pi = \frac{7\pi}{4} + 2\pi$ will also provide a basis for finding solutions for *x*.
$\frac{7\pi}{4} + 2\pi = \frac{7\pi}{4} + \frac{8\pi}{4} = \frac{15\pi}{4}$. Setting $3x = \frac{15\pi}{4}$ and solving for *x*, we have $x = \frac{15\pi}{12} = \frac{5\pi}{4}$.
Perhaps $\theta + 4\pi = \frac{7\pi}{4} + 4\pi$ will also provide a basis for finding solutions for *x*.
$\frac{7\pi}{4} + 4\pi = \frac{7\pi}{4} + \frac{16\pi}{4} = \frac{23\pi}{4}$. Setting $3x = \frac{23\pi}{4}$ and solving for *x*, we have $x = \frac{23\pi}{12}$.
Perhaps $\theta + 6\pi = \frac{7\pi}{4} + 6\pi$ will also provide a basis for finding solutions for *x*.
$\frac{7\pi}{4} + 6\pi = \frac{7\pi}{4} + \frac{24\pi}{4} = \frac{31\pi}{4}$. Setting $3x = \frac{31\pi}{4}$ and solving for *x*, we have $x = \frac{31\pi}{12}$.
But $\frac{31\pi}{12} > \frac{24\pi}{12} = 2\pi$ is outside the interval $[0,2\pi)$.

Therefore, the only solutions for *x* are $\frac{5\pi}{12}, \frac{7\pi}{12}, \frac{13\pi}{12}, \frac{15\pi}{12}, \frac{21\pi}{12}, \frac{23\pi}{12}$.

Check: $\csc\left(3 \cdot \frac{5\pi}{12}\right) + \sqrt{2} = \csc\left(\frac{5\pi}{4}\right) + \sqrt{2} = \frac{1}{\sin\left(\frac{5\pi}{4}\right)} + \sqrt{2} = \frac{1}{\left(\frac{-1}{\sqrt{2}}\right)} + \sqrt{2} = 0.$ ✓

$\csc\left(3 \cdot \frac{7\pi}{12}\right) + \sqrt{2} = \csc\left(\frac{7\pi}{4}\right) + \sqrt{2} = \frac{1}{\sin\left(\frac{7\pi}{4}\right)} + \sqrt{2} = \frac{1}{\left(\frac{-1}{\sqrt{2}}\right)} + \sqrt{2} = -\sqrt{2} + \sqrt{2} = 0.$ ✓

$\csc\left(3 \cdot \frac{13\pi}{12}\right) + \sqrt{2} = \csc\left(\frac{13\pi}{4}\right) + \sqrt{2} = \frac{1}{\sin\left(\frac{13\pi}{4}\right)} + \sqrt{2} = \frac{1}{\sin\left(\frac{5\pi}{4}\right)} + \sqrt{2} = \frac{1}{\left(\frac{-1}{\sqrt{2}}\right)} + \sqrt{2} = 0.$ ✓

$\csc\left(3\cdot\frac{15\pi}{12}\right)+\sqrt{2}=\csc\left(\frac{15\pi}{4}\right)+\sqrt{2}=\frac{1}{\sin\left(\frac{15\pi}{4}\right)}+\sqrt{2}=\frac{1}{\sin\left(\frac{7\pi}{4}\right)}+\sqrt{2}=\frac{1}{\left(\frac{-1}{\sqrt{2}}\right)}+\sqrt{2}=0.$ ✓

$\csc\left(3\cdot\frac{21\pi}{12}\right)+\sqrt{2}=\csc\left(\frac{21\pi}{4}\right)+\sqrt{2}=\frac{1}{\sin\left(\frac{21\pi}{4}\right)}+\sqrt{2}=\frac{1}{\sin\left(\frac{5\pi}{4}\right)}+\sqrt{2}=\frac{1}{\left(\frac{-1}{\sqrt{2}}\right)}+\sqrt{2}=0.$ ✓

$\csc\left(3\cdot\frac{23\pi}{12}\right)+\sqrt{2}=\csc\left(\frac{23\pi}{4}\right)+\sqrt{2}=\frac{1}{\sin\left(\frac{23\pi}{4}\right)}+\sqrt{2}=\frac{1}{\sin\left(\frac{7\pi}{4}\right)}+\sqrt{2}=\frac{1}{\left(\frac{-1}{\sqrt{2}}\right)}+\sqrt{2}=0.$ ✓

8. $\sec(5x)+1=0 \quad\Rightarrow\quad \sec(5x)=-1 \quad\Rightarrow\quad \frac{1}{\cos(5x)}=-1 \quad\Rightarrow\quad \cos(5x)=-1.$
According to *WR* no. 266, problem 19, $\cos(\theta)=-1$ for $\theta=\pi$.

Regarding $\theta=\pi$, set $5x=\theta$. Hence,$5x=\pi$. Solving for *x*, we have $x=\frac{\pi}{5}$.
Perhaps $\theta+2\pi=\pi+2\pi$ will also provide a basis for finding solutions for *x*.
$\pi+2\pi=3\pi$. Setting $5x=3\pi$ and solving for *x*, we have $x=\frac{3\pi}{5}$.
Perhaps $\theta+4\pi=\pi+4\pi$ will also provide a basis for finding solutions for *x*.
$\pi+4\pi=5\pi$. Setting $5x=5\pi$ and solving for *x*, we have $x=\frac{5\pi}{5}=\pi$.
Perhaps $\theta+6\pi=\pi+6\pi$ will also provide a basis for finding solutions for *x*.
$\pi+6\pi=7\pi$. Setting $5x=7\pi$ and solving for *x*, we have $x=\frac{7\pi}{5}$.
Perhaps $\theta+8\pi=\pi+8\pi$ will also provide a basis for finding solutions for *x*.
$\pi+8\pi=9\pi$. Setting $5x=9\pi$ and solving for *x*, we have $x=\frac{9\pi}{5}$.
Perhaps $\theta+10\pi=\pi+10\pi$ will also provide a basis for finding solutions for *x*.
$\pi+10\pi=11\pi$. Setting $5x=11\pi$ and solving for *x*, we have $x=\frac{11\pi}{5}$.
But $\frac{11\pi}{5}>\frac{10\pi}{5}=2\pi$ is outside the interval $[0,2\pi)$.

Therefore, the only solutions for *x* are $\frac{\pi}{5},\frac{3\pi}{5},\frac{5\pi}{5},\frac{7\pi}{5},\frac{9\pi}{5},\frac{11\pi}{5}$.

Check: $\csc\left(\frac{\pi}{2}\right)-1=\frac{1}{\sin\left(\frac{\pi}{2}\right)}-1=\frac{1}{1}-1=1-1=0.$✓

9. $\csc\left(\frac{x}{2}\right)-1=0 \quad\Rightarrow\quad \csc\left(\frac{x}{2}\right)=1 \quad\Rightarrow\quad \frac{1}{\sin\left(\frac{x}{2}\right)}=1 \quad\Rightarrow\quad \sin\left(\frac{x}{2}\right)=1.$
According to *WR* no. 266, problem 13, $\sin(\theta)=1$ for $\theta=\frac{\pi}{2}$.

Regarding $\theta=\frac{\pi}{2}$, set $\frac{x}{2}=\theta$. Hence, $\frac{x}{2}=\frac{\pi}{2}$. Solving for *x*, we have $x=\frac{2\pi}{2}=\pi$. Perhaps $\theta+2\pi=\frac{\pi}{2}+2\pi$ will also provide a basis for finding solutions for *x*.
$\frac{\pi}{2}+2\pi=\frac{\pi}{2}+\frac{4\pi}{2}=\frac{5\pi}{2}$. Setting $\frac{x}{2}=\frac{5\pi}{2}$ and solving for *x*, we have $x=\frac{10\pi}{2}=5\pi$.
But $5\pi>2\pi$ is outside the interval $[0,2\pi)$.

Therefore, the only solution for *x* is π.

Check: $\csc\left(\frac{\pi}{2}\right)-1=\frac{1}{\sin\left(\frac{\pi}{2}\right)}-1=\frac{1}{1}-1=1-1=0.$✓