The Weekly Rigor

No. 280 "A mathematician is a machine for turning coffee into theorems." November 2, 2019

12 Problems Solving Composite Trigonometric Equations (Type III)
(Part 2)

SELECTED SOLUTIONS

1. $3\tan(2x) - \sqrt{3} = 0 \quad \Longrightarrow \quad \tan(2x) = \frac{\sqrt{3}}{3} = \frac{\sqrt{3}}{3} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{3}{3\sqrt{3}} = \frac{1}{\sqrt{3}}$.
According to *WR* no. 272, problem 1, $\tan(\theta) = \frac{1}{\sqrt{3}}$ for $\theta = \frac{\pi}{6}$ and $\theta = \frac{7\pi}{6}$.

Regarding $\theta = \frac{\pi}{6}$, set $2x = \theta$. Hence, $2x = \frac{\pi}{6}$. Solving for *x*, we have $x = \frac{\pi}{12}$.
Perhaps $\theta + 2\pi = \frac{\pi}{6} + 2\pi$ will also provide a basis for finding solutions for *x*.
$\frac{\pi}{6} + 2\pi = \frac{\pi}{6} + \frac{12\pi}{6} = \frac{13\pi}{6}$. Setting $2x = \frac{13\pi}{6}$ and solving for *x*, we have $x = \frac{13\pi}{12}$.
Perhaps $\theta + 4\pi = \frac{\pi}{6} + 4\pi$ will also provide a basis for finding solutions for *x*.
$\frac{\pi}{6} + 4\pi = \frac{\pi}{6} + \frac{24\pi}{6} = \frac{25\pi}{6}$. Setting $2x = \frac{25\pi}{6}$ and solving for *x*, we have $x = \frac{25\pi}{12}$.
But $\frac{25\pi}{12} > \frac{24\pi}{12} = 2\pi$ is outside the interval $[0,2\pi)$.

Regarding $\theta = \frac{7\pi}{6}$, set $2x = \theta$. Hence, $2x = \frac{7\pi}{6}$. Solving for *x*, we have $x = \frac{7\pi}{12}$.
Perhaps $\theta + 2\pi = \frac{7\pi}{6} + 2\pi$ will also provide a basis for finding solutions for *x*.
$\frac{7\pi}{6} + 2\pi = \frac{7\pi}{6} + \frac{12\pi}{6} = \frac{19\pi}{6}$. Setting $2x = \frac{19\pi}{6}$ and solving for *x*, we have $x = \frac{19\pi}{12}$.
Perhaps $\theta + 4\pi = \frac{7\pi}{6} + 4\pi$ will also provide a basis for finding solutions for *x*.
$\frac{7\pi}{6} + 4\pi = \frac{7\pi}{6} + \frac{24\pi}{6} = \frac{31\pi}{6}$. Setting $2x = \frac{31\pi}{6}$ and solving for *x*, we have $x = \frac{31\pi}{12}$.
But $\frac{31\pi}{12} > \frac{24\pi}{12} = 2\pi$ is outside the interval $[0,2\pi)$.

Therefore, the only solutions for *x* are $\frac{\pi}{12}, \frac{7\pi}{12}, \frac{13\pi}{12}, \frac{19\pi}{12}$.

Check: $3\tan\left(2 \cdot \frac{\pi}{12}\right) - \sqrt{3} = 3\tan\left(\frac{\pi}{6}\right) - \sqrt{3} = 3\frac{1}{\sqrt{3}} - \sqrt{3} = \sqrt{3} - \sqrt{3} = 0$. ✓
$3\tan\left(2 \cdot \frac{7\pi}{12}\right) - \sqrt{3} = 3\tan\left(\frac{7\pi}{6}\right) - \sqrt{3} = 3\frac{1}{\sqrt{3}} - \sqrt{3} = \sqrt{3} - \sqrt{3} = 0$. ✓
$3\tan\left(2 \cdot \frac{13\pi}{12}\right) - \sqrt{3} = 3\tan\left(\frac{13\pi}{6}\right) - \sqrt{3} = 3\tan\left(\frac{\pi}{6}\right) - \sqrt{3} = 3\frac{1}{\sqrt{3}} - \sqrt{3} = \sqrt{3} - \sqrt{3} = 0$. ✓
$3\tan\left(2 \cdot \frac{19\pi}{12}\right) - \sqrt{3} = 3\tan\left(\frac{19\pi}{6}\right) - \sqrt{3} = 3\tan\left(\frac{7\pi}{6}\right) - \sqrt{3} = 3\frac{1}{\sqrt{3}} - \sqrt{3} = \sqrt{3} - \sqrt{3} = 0$. ✓

4. $\tan^2\left(\frac{x}{3}\right) - 3 = 0 \quad \Longrightarrow \quad \tan^2\left(\frac{x}{3}\right) = 3 \quad \Longrightarrow \quad \tan\left(\frac{x}{3}\right) = \pm\sqrt{3} = \pm\frac{\sqrt{3}}{1}.$

According to *WR* no. 272, problem 19, $\tan(\theta) = \pm\frac{\sqrt{3}}{1}$ for $\theta = \frac{\pi}{3}$, $\theta = \frac{2\pi}{3}$, $\theta = \frac{4\pi}{3}$, and $\theta = \frac{5\pi}{3}$.

Regarding $\theta = \frac{\pi}{3}$, set $\frac{x}{3} = \theta$. Hence, $\frac{x}{3} = \frac{\pi}{3}$. Solving for *x*, we have $x = \pi$.
Perhaps $\theta + 2\pi = \frac{\pi}{3} + 2\pi$ will also provide a basis for finding solutions for *x*.
$\frac{\pi}{3} + 2\pi = \frac{\pi}{3} + \frac{6\pi}{3} = \frac{7\pi}{3}$. Setting $\frac{x}{3} = \frac{7\pi}{3}$ and solving for *x*, we have $x = 7\pi$.
But $7\pi > 2\pi$ is outside the interval $[0,2\pi)$.

Regarding $\theta = \frac{2\pi}{3}$, set $\frac{x}{3} = \theta$. Hence, $\frac{x}{3} = \frac{2\pi}{3}$. Solving for *x*, we have $x = 2\pi$.
But 2π is outside the interval $[0,2\pi)$.

Regarding $\theta = \frac{4\pi}{3}$, set $\frac{x}{3} = \theta$. Hence, $\frac{x}{3} = \frac{4\pi}{3}$. Solving for *x*, we have $x = 4\pi$.
But 4π is outside the interval $[0,2\pi)$.

Regarding $\theta = \frac{5\pi}{3}$, set $\frac{x}{3} = \theta$. Hence, $\frac{x}{3} = \frac{5\pi}{3}$. Solving for *x*, we have $x = 5\pi$.
But 5π is outside the interval $[0,2\pi)$.

Therefore, the only solution for *x* is π.

Check: $\tan^2\left(\frac{\pi}{3}\right) - 3 = \left(\frac{\sqrt{3}}{1}\right)^2 - 3 = \left(\sqrt{3}\right)^2 - 3 = 3 - 3 = 0.$ ✓