The Weekly Rigor

No. 281 "A mathematician is a machine for turning coffee into theorems." November 9, 2019

12 Problems Solving Composite Trigonometric Equations (Type III)
(Part 3)

5. $\tan(4x) - 1 = 0 \quad \Longrightarrow \quad \tan(4x) = 1$. According to *WR* no. 273, problem 21, $\tan(\theta) = 1$ for $\theta = \frac{\pi}{4}$ and $\theta = \frac{5\pi}{4}$.

Regarding $\theta = \frac{\pi}{4}$, set $4x = \theta$. Hence, $4x = \frac{\pi}{4}$. Solving for *x*, we have $x = \frac{\pi}{16}$.
Perhaps $\theta + 2\pi = \frac{\pi}{4} + 2\pi$ will also provide a basis for finding solutions for *x*.
$\frac{\pi}{4} + 2\pi = \frac{\pi}{4} + \frac{8\pi}{4} = \frac{9\pi}{4}$. Setting $4x = \frac{9\pi}{4}$ and solving for *x*, we have $x = \frac{9\pi}{16}$.
Perhaps $\theta + 4\pi = \frac{\pi}{4} + 4\pi$ will also provide a basis for finding solutions for *x*.
$\frac{\pi}{4} + 4\pi = \frac{\pi}{4} + \frac{16\pi}{4} = \frac{17\pi}{4}$. Setting $4x = \frac{17\pi}{4}$ and solving for *x*, we have $x = \frac{17\pi}{16}$.
Perhaps $\theta + 6\pi = \frac{\pi}{4} + 6\pi$ will also provide a basis for finding solutions for *x*.
$\frac{\pi}{4} + 6\pi = \frac{\pi}{4} + \frac{24\pi}{4} = \frac{25\pi}{4}$. Setting $4x = \frac{25\pi}{4}$ and solving for *x*, we have $x = \frac{25\pi}{16}$.
Perhaps $\theta + 8\pi = \frac{\pi}{4} + 8\pi$ will also provide a basis for finding solutions for *x*.
$\frac{\pi}{4} + 8\pi = \frac{\pi}{4} + \frac{32\pi}{4} = \frac{33\pi}{4}$. Setting $4x = \frac{33\pi}{4}$ and solving for *x*, we have $x = \frac{33\pi}{16}$.
But $\frac{33\pi}{16} > \frac{32\pi}{16} = 2\pi$ is outside the interval $[0, 2\pi)$.

Regarding $\theta = \frac{5\pi}{4}$, set $4x = \theta$. Hence, $4x = \frac{5\pi}{4}$. Solving for *x*, we have $x = \frac{5\pi}{16}$.
Perhaps $\theta + 2\pi = \frac{5\pi}{4} + 2\pi$ will also provide a basis for finding solutions for *x*.
$\frac{5\pi}{4} + 2\pi = \frac{5\pi}{4} + \frac{8\pi}{4} = \frac{13\pi}{4}$. Setting $4x = \frac{13\pi}{4}$ and solving for *x*, we have $x = \frac{13\pi}{16}$.
Perhaps $\theta + 4\pi = \frac{5\pi}{4} + 4\pi$ will also provide a basis for finding solutions for *x*.
$\frac{5\pi}{4} + 4\pi = \frac{5\pi}{4} + \frac{16\pi}{4} = \frac{21\pi}{4}$. Setting $4x = \frac{21\pi}{4}$ and solving for *x*, we have $x = \frac{21\pi}{16}$.
Perhaps $\theta + 6\pi = \frac{5\pi}{4} + 6\pi$ will also provide a basis for finding solutions for *x*.
$\frac{5\pi}{4} + 6\pi = \frac{5\pi}{4} + \frac{24\pi}{4} = \frac{29\pi}{4}$. Setting $4x = \frac{29\pi}{4}$ and solving for *x*, we have $x = \frac{29\pi}{16}$.
Perhaps $\theta + 8\pi = \frac{5\pi}{4} + 8\pi$ will also provide a basis for finding solutions for *x*.
$\frac{5\pi}{4} + 8\pi = \frac{5\pi}{4} + \frac{32\pi}{4} = \frac{37\pi}{4}$. Setting $4x = \frac{37\pi}{4}$ and solving for *x*, we have $x = \frac{37\pi}{16}$.
But $\frac{37\pi}{16} > \frac{32\pi}{16} = 2\pi$ is outside the interval $[0, 2\pi)$.

Therefore, the only solutions for *x* are $\frac{\pi}{16}, \frac{5\pi}{16}, \frac{9\pi}{16}, \frac{13\pi}{16}, \frac{17\pi}{16}, \frac{21\pi}{16}, \frac{25\pi}{16}, \frac{29\pi}{16}$.

Check: $\tan\left(4\cdot\frac{\pi}{16}\right)-1=\tan\left(\frac{\pi}{4}\right)-1=1-1=0.$ ✓

$\tan\left(4\cdot\frac{5\pi}{16}\right)-1=\tan\left(\frac{5\pi}{4}\right)-1=1-1=0.$ ✓

$\tan\left(4\cdot\frac{9\pi}{16}\right)-1=\tan\left(\frac{9\pi}{4}\right)-1=\tan\left(\frac{\pi}{4}\right)-1=1-1=0.$ ✓

$\tan\left(4\cdot\frac{13\pi}{16}\right)-1=\tan\left(\frac{13\pi}{4}\right)-1=\tan\left(\frac{5\pi}{4}\right)-1=1-1=0.$ ✓

$\tan\left(4\cdot\frac{17\pi}{16}\right)-1=\tan\left(\frac{17\pi}{4}\right)-1=\tan\left(\frac{9\pi}{4}\right)-1=\tan\left(\frac{\pi}{4}\right)-1=1-1=0.$ ✓

$\tan\left(4\cdot\frac{21\pi}{16}\right)-1=\tan\left(\frac{21\pi}{4}\right)-1=\tan\left(\frac{13\pi}{4}\right)-1=\tan\left(\frac{5\pi}{4}\right)-1=1-1=0.$ ✓

$\tan\left(4\cdot\frac{25\pi}{16}\right)-1=\tan\left(\frac{25\pi}{4}\right)-1=\tan\left(\frac{17\pi}{4}\right)-1=\tan\left(\frac{9\pi}{4}\right)-1=1-1=0.$ ✓

$\tan\left(4\cdot\frac{29\pi}{16}\right)-1=\tan\left(\frac{29\pi}{4}\right)-1=\tan\left(\frac{21\pi}{4}\right)-1=\tan\left(\frac{13\pi}{4}\right)-1=1-1=0.$ ✓

10. $\cot^2\left(\frac{1}{2}x\right)-3=0 \quad\Rightarrow\quad \cot^2\left(\frac{1}{2}x\right)=3 \quad\Rightarrow\quad \frac{1}{\tan^2\left(\frac{1}{2}x\right)}=3 \quad\Rightarrow$

$\Rightarrow\quad \tan^2\left(\frac{1}{2}x\right)=\frac{1}{3} \quad\Rightarrow\quad \tan\left(\frac{1}{2}x\right)=\pm\frac{1}{\sqrt{3}}$. According to *WR* no. 273, problem 26, $\tan(\theta)=\pm\frac{1}{\sqrt{3}}$ for $\theta=\frac{\pi}{6},\frac{5\pi}{6},\frac{7\pi}{6}$, and $\frac{11\pi}{6}$.

Regarding $\theta=\frac{\pi}{6}$, set $\frac{1}{2}x=\theta$. Hence, $\frac{1}{2}x=\frac{\pi}{6}$. Solving for *x*, we have $x=\frac{\pi}{3}$. Perhaps $\theta+2\pi=\frac{\pi}{6}+2\pi$ will also provide a basis for finding solutions for *x*. $\frac{\pi}{6}+2\pi=\frac{\pi}{6}+\frac{12\pi}{6}=\frac{13\pi}{6}$. Setting $\frac{1}{2}x=\frac{13\pi}{6}$ and solving for *x*, we have $x=\frac{13\pi}{3}$. But $\frac{13\pi}{3}>\frac{6\pi}{3}=2\pi$ is outside the interval $[0,2\pi)$.

Regarding $\theta=\frac{5\pi}{6}$, set $\frac{1}{2}x=\theta$. Hence, $\frac{1}{2}x=\frac{5\pi}{6}$. Solving for *x*, we have $x=\frac{5\pi}{3}$. Perhaps $\theta+2\pi=\frac{5\pi}{6}+2\pi$ will also provide a basis for finding solutions for *x*. $\frac{5\pi}{6}+2\pi=\frac{5\pi}{6}+\frac{12\pi}{6}=\frac{17\pi}{6}$. Setting $\frac{1}{2}x=\frac{17\pi}{6}$ and solving for *x*, we have $x=\frac{17\pi}{3}$. But $\frac{17\pi}{3}>\frac{6\pi}{3}=2\pi$ is outside the interval $[0,2\pi)$.

Regarding $\theta=\frac{7\pi}{6}$, set $\frac{1}{2}x=\theta$. Hence, $\frac{1}{2}x=\frac{7\pi}{6}$. Solving for *x*, we have $x=\frac{7\pi}{3}$. But $\frac{7\pi}{3}>\frac{6\pi}{3}=2\pi$ is outside the interval $[0,2\pi)$.

Regarding $\theta=\frac{11\pi}{6}$, set $\frac{1}{2}x=\theta$. Hence, $\frac{1}{2}x=\frac{11\pi}{6}$. Solving for *x*, we have $x=\frac{11\pi}{3}$. But $\frac{11\pi}{3}>\frac{6\pi}{3}=2\pi$ is outside the interval $[0,2\pi)$.

Therefore, the only solutions for *x* are $\frac{\pi}{3}$, $\frac{5\pi}{3}$.

Check: $\cot^2\left(\frac{1}{2}\cdot\frac{\pi}{3}\right)-3=\cot^2\left(\frac{\pi}{6}\right)-3=\frac{1}{\tan^2\left(\frac{\pi}{6}\right)}-3=\left(\sqrt{3}\right)^2-3=3-3=0.$ ✓

$\cot^2\left(\frac{1}{2}\cdot\frac{5\pi}{3}\right)-3=\cot^2\left(\frac{5\pi}{6}\right)-3=\frac{1}{\tan^2\left(\frac{5\pi}{6}\right)}-3=\left(\sqrt{3}\right)^2-3=3-3=0.$ ✓