

The Weekly Rigor

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“A mathematician is a machine for turning coffee into theorems.”

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Verifying Trigonometric Identities with Simple Arguments Involving the Product of Two Trigonometric Functions: Problems with Solutions (Part 1)

INTRODUCTION

This set of problems depend on the following trigonometric identities as given:

$$\sec \theta = \frac{1}{\cos \theta}, \quad \csc \theta = \frac{1}{\sin \theta}, \quad \cot \theta = \frac{1}{\tan \theta}. \quad (\text{Reciprocal Identities})$$

$$\tan \theta = \frac{\sin \theta}{\cos \theta}, \quad \cot \theta = \frac{\cos \theta}{\sin \theta}. \quad (\text{Ratio Identities})$$

$$\sin^2 \theta + \cos^2 \theta = 1.$$

$$\tan^2 \theta + 1 = \sec^2 \theta. \quad (\text{Pythagorean Identities})$$

$$1 + \cot^2 \theta = \csc^2 \theta.$$

Strictly speaking, the identity $\cot \theta = \frac{\cos \theta}{\sin \theta}$ should be regarded as a corollary of some of the other identities, viz.,

$$\cot \theta = \frac{1}{\tan \theta} = \frac{1}{\left(\frac{\sin \theta}{\cos \theta}\right)} = \frac{1}{1} \cdot \frac{\cos \theta}{\sin \theta} = \frac{\cos \theta}{\sin \theta}.$$

Similarly, the second and third Pythagorean Identities follow from the first. To wit:

$$\tan^2 \theta + 1 = \frac{\sin^2 \theta}{\cos^2 \theta} + 1 = \frac{\sin^2 \theta}{\cos^2 \theta} + \frac{\cos^2 \theta}{\cos^2 \theta} = \frac{\sin^2 \theta + \cos^2 \theta}{\cos^2 \theta} = \frac{1}{\cos^2 \theta} = \sec^2 \theta, \text{ and}$$

$$1 + \cot^2 \theta = 1 + \frac{\cos^2 \theta}{\sin^2 \theta} = \frac{\sin^2 \theta}{\sin^2 \theta} + \frac{\cos^2 \theta}{\sin^2 \theta} = \frac{\sin^2 \theta + \cos^2 \theta}{\sin^2 \theta} = \frac{1}{\sin^2 \theta} = \csc^2 \theta.$$

Additional useful corollaries are the following:

$$\cos \theta = \frac{1}{\sec \theta}, \quad \sin \theta = \frac{1}{\csc \theta}, \quad \tan \theta = \frac{1}{\cot \theta}.$$

Proofs: $\frac{1}{\sec \theta} = \frac{1}{\left(\frac{1}{\cos \theta}\right)} = \frac{1}{1} \cdot \frac{\cos \theta}{1} = \cos \theta.$

$$\frac{1}{\csc \theta} = \frac{1}{\left(\frac{1}{\sin \theta}\right)} = \frac{1}{1} \cdot \frac{\sin \theta}{1} = \sin \theta.$$

$$\frac{1}{\cot \theta} = \frac{1}{\left(\frac{1}{\tan \theta}\right)} = \frac{1}{1} \cdot \frac{\tan \theta}{1} = \tan \theta.$$

A last set of such useful corollaries are the following:

$$\sin^2 \theta = 1 - \cos^2 \theta, \quad \cos^2 \theta = 1 - \sin^2 \theta, \quad \tan^2 \theta = \sec^2 \theta - 1.$$

$$1 = \sec^2 \theta - \tan^2 \theta, \quad 1 = \csc^2 \theta - \cot^2 \theta, \quad \cot^2 \theta = \csc^2 \theta - 1.$$

Proofs: $1 - \cos^2 \theta = (\sin^2 \theta + \cos^2 \theta) - \cos^2 \theta = \sin^2 \theta.$

$$1 - \sin^2 \theta = (\sin^2 \theta + \cos^2 \theta) - \sin^2 \theta = \cos^2 \theta.$$

$$\sec^2 \theta - 1 = (\tan^2 \theta + 1) - 1 = \tan^2 \theta.$$

$$\sec^2 \theta - \tan^2 \theta = (\tan^2 \theta + 1) - \tan^2 \theta = 1.$$

$$\csc^2 \theta - \cot^2 \theta = (1 + \cot^2 \theta) - \cot^2 \theta = 1.$$

$$\csc^2 \theta - 1 = (1 + \cot^2 \theta) - 1 = \cot^2 \theta.$$

“Only he who never plays, never loses.”